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Transitivity with quasi-isolated point in G-space

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Abstract

Some chaos characterizations on G-space L are investigate in this paper, like : (open-set, orbit, strictly orbit,)G-transitive. We find some results related between them without isolated point in G-space L.

Keywords: G-transitive, G-orbit transitive, open –set G-transitive, G - ω-transitive, , G -transitive point, isolated point, quasi isolated point.

1. INTRODUCTION

One of the leading areas of researches in Mathematics.The dynamical systems is. Many authors across the globe are contributing towards the researches on properties of dynamical systems and chaotic behavior of maps in topological dynamical systems. These researches closely related with many branches of applications in sciences. In the past, A special class of dynamical systems known as chaotic dynamical systems has been studied in detail[3].

The authors in [8], study one of the most significant properties of dynamical systems named topological transitivity. Orbit-transitivity, strictly orbit-transitivity and ω-transitivity are related or are equivalent to the transitivity in topological spaces have been studied see [1]and [2],with quasi-isolated point or without quasi-isolated point.

Dynamical properties of group actions have been defined and studied in detail with other stronger properties [11].And, chaotic properties are studied on topological group. The chaotic dynamical concepts are generalized on topological group [10]. So in [9] the authors studied the relations between (orbit, strictly orbit, open –set and)G-transitive. In this paper we show these relations without quasi isolated point.

2. Basic Definitions

Definition 2.1: [4]

Let $\hbar_{\mathfrak{g}}: \mathcal{L} \to \mathcal{L}$ be a map on a $\mathbb{G}-$space $\mathcal{L}$. The map $\hbar_g$ is $\mathbb{G}-$transitive (simply $\mathbb{G}-T$) if for every two open subsets $\mathcal{M}$ and $\mathcal{N}$, non-empty on $\mathcal{L}$, there is $k \in \mathbb{Z}$ such that

$$\hbar_{\mathfrak{g}}^{k}(\mathcal{M}) \cap \mathcal{N} \neq \emptyset, \mathfrak{g} \in \mathbb{G}.$$

Definition 2. 2: [7]

Let $\hbar_{\mathfrak{g}}: \mathcal{L} \to \mathcal{L}$ be a map on a $\mathbb{G}-$space $\mathcal{L}$. The map $\hbar_{\mathfrak{g}}$ is $\mathbb{G}-$orbit transitive (simply $\mathbb{G}-OT$) if there is $\ell \in \mathcal{L}$such that $\overline{\mathbb{G}\,(\ell)} = \mathcal{L}$.

Definition 2.3: [4]

Let $\hbar_{\mathfrak{g}}: \mathcal{L} \to \mathcal{L}$ be a map on a $\mathbb{G}-$space $\mathcal{L}$. A point $\ell \in \mathcal{L}$ is $\mathbb{G}-$transitive point of the map $\hbar_{\mathfrak{g}}$ if its $\mathbb{G}-$ orbit, $\mathbb{G}(\ell)$, is dense in $\mathcal{L}$.

Definition 2.4: [7]

Let $\hbar_{\mathrm{g}}: \mathcal{L} \to \mathcal{L}$ be a map on a $\mathbb{G}-$ space $\mathcal{L}$. The map $\hbar_{\mathrm{g}}$ is open –set $\mathbb{G}-$transitive (simply open set $\mathbb{G}-T$) if for every two open subsets $\mathcal{M}$ and $\mathcal{N}$, non-empty on $\mathcal{L}$ there is $k \in \mathbb{N}$ such that

$$\hbar_{\mathrm{g}}^{k}(\mathcal{M}) \cap \mathcal{N} \neq \emptyset, \mathrm{g} \in \mathbb{G}.$$

Definition 2.5:[6]

Let $\hbar_{\mathrm{g}}: \mathcal{L} \to \mathcal{L}$ be a map on a $\mathbb{G}-$space $\mathcal{L}$. The map $\hbar_{\mathrm{g}}$ is $\mathbb{G}$ $\omega-$transitive (simply $\mathbb{G}-\omega-T$)if $\omega_{\mathrm{G}}\left(\mathcal{L}, \hbar_{\mathrm{g}}\right) = \mathcal{L}$, for some $\ell \in \mathcal{L}$.

Definition 2.6:[9]

Let $\hbar_{\mathrm{g}}: \mathcal{L} \to \mathcal{L}$ be a map on a $\mathbb{G}-$space $\mathcal{L}$. The map $\hbar_{\mathrm{g}}$ is strictly orbit $\mathbb{G}-$ transitive (simply $\mathbb{G}-SO$) if there is a point $\ell \in \mathcal{L}$ and such that

$$\overline{\mathbb{G}(\hbar(\ell))} = \mathcal{L}, \mathrm{g} \in \mathbb{G}.$$

And we generalized the definitions of isolated point and quasi isolated point in $\mathbb{G}-$space as we see below :

Definition 2.7:

Let $\hbar_{\mathrm{g}}: \mathcal{L} \to \mathcal{L}$ be a map on a $\mathbb{G}-$space $\mathcal{L}$. A point $\ell \in \mathcal{A}$ is an isolated point (simply iso.pt.) of $\mathcal{A}$ if there exists a neighborhood $\mathcal{M}$ of ℓ in $\mathbb{G}-$space $\mathcal{L}$such that $\mathcal{M} \cap \mathcal{A} = \{\ell\}$.

Definition 2.8:

Let $\hbar_{\mathrm{g}}: \mathcal{L} \to \mathcal{L}$ be a map on a $\mathbb{G}-$space $\mathcal{L}$. A point ℓ is a quasi – isolated point (simply qu.iso.pt.) of $\mathcal{L}$ if there exist a subset $\mathcal{A}$ of $\mathcal{L}$ which is dense such that $\ell \in \mathcal{A}$ and ℓ is iso.pt. in $\mathcal{A}$.

Remark* 2.9*: From definition 2.7 and 2.8 one can see that every iso.pt. of $\mathcal{L}$ is qu.iso.pt..

3. Main Results

Lemma* 3.1*:

Let $\mathcal{L}$be a $\mathbb{G}$-space, then a point ℓ is a qu.iso.pt. of $\mathcal{L}$ if and only if $int\,(\overline{\{\ell\}}) \neq \emptyset$.

<u>Proof</u> : $\Longrightarrow$ Let ℓ is a qu.iso.pt. of $\mathbb{G}$-space $\mathcal{L}$, then there exists a subset $\mathcal{A}$ which is dense in $\mathcal{L}$ and $\mathcal{M}$ which is open neighborhood of ℓ such that $\mathcal{M} \cap \mathcal{A} = \{\ell\}$. since $\mathcal{A}$ is dense, then

$$\overline{\{\ell\}} \cup \overline{\mathcal{A}-\{\ell\}} = \overline{\mathcal{A}} = \mathcal{L},$$

And

$$\overline{A-\{\ell\}} = \overline{\mathcal{A}-\mathcal{M}} \subset \mathcal{L} - \mathcal{M},$$

we have $\mathcal{M} \subset \overline{\{\ell\}}$ and hence

$$\mathcal{M} \subset int(\overline{\{\ell\}}) \neq \emptyset.$$

That means $int(\overline{\{\ell\}}) \neq \emptyset$.

$\Longleftarrow$ Write $\mathcal{N} = int(\overline{\{\ell\}})$. If $\mathcal{N} \neq \emptyset$, then there exist $\ell \in \mathcal{N}$. let

$$\mathcal{A} = \{\ell\} \cup (\mathcal{L} - \mathcal{N}). \qquad (1.1)$$

then by definition of iso.pt. ℓ is iso.pt. of $\mathcal{A}$. by (1.1) then

$$\overline{\mathcal{A}} = \overline{\{\ell\}} \cup \overline{\mathcal{L}-\mathcal{N}} \supset \mathcal{N} \cup (\mathcal{L} - \mathcal{N}) = \mathcal{L},$$

So $\mathcal{A}$ is dense in $\mathcal{L}$. hence $\mathcal{L} \subset \overline{\mathcal{A}}$, by definition of qu.iso.pt. then ℓ is qu.iso.pt. of $\mathcal{L}$.

■

Lemma 3.2:

Let $\mathcal{L}$ be a $\mathbb{G}-$ space, then $\mathcal{L}$ is without qu.iso.pts. if and only if for all subset $\mathcal{M}$ which are non-empty and open of $\mathcal{L}$, and all subset which are dense of $\mathcal{M}$ has two points at least.

<u>Proof</u> : $\Longrightarrow$Let $\mathcal{L}$ has no qu.iso.pt. ℓ, then by lemma 3.1, the set $\{\ell\}$ consists of one point is a dense subset of $int\left(\overline{\{\ell\}}\right)$ which is non-empty and open set .

If there exists a subset $\mathcal{M}$of $\mathcal{L}$ such that $\Longleftarrow$ non- empty and open , and a point $\ell \in \mathcal{M}$such that $\{\ell\}$ is dense in $\mathcal{M}$, then by lemma 3.1, ℓ is qu.iso.pt. of $\mathcal{L}$.

■

Lemma 3.3:

Let $\mathcal{L}$ be a $\mathbb{G}-$space. Then a point $\ell \in \mathcal{L}$ is an iso.pt. of $\mathcal{L}$ if and only if ℓ is a qu.iso.pt. of $\mathcal{L}$.

Proof : $\Longrightarrow$ Suppose that ℓ is iso.pt., then by remark 2.9, ℓ is qu.iso.pt..

$\Longleftarrow$ assume that ℓ is a qu.iso.pt. . By lemma 3.1, then

$$Int(\overline{\{\ell\}}) \neq \phi.$$

Since $\mathcal{L}$ is $\mathbb{G}$- space , one has $\overline{\{\ell\}} = \{\ell\}$. thus

$$\{\ell\} = Int(\overline{\{\ell\}})$$

is open set, thus $\{\ell\} = Int(\overline{\{\ell\}})$ is an open set, thus

$$\overline{\{\ell\}} = \mathrm{Int}(\overline{\{\ell\}}),$$

then $\overline{\{\ell\}}$ is open set, so

$$\overline{\{\ell\}} \cap \{\ell\} = \ell.$$

hence ℓ is an iso.pt. of $\mathcal{L}$.

■

***Lemma* 3.4:**

Let $\mathcal{L}$ be $\mathbb{G}-$space without qu.iso.pt. then for any given $\ell \in \mathcal{L}$, $\hbar_g(\ell)$ is $\mathbb{G}-SO$ of $\hbar_g$ if and only if $\hbar_g^n(\ell)$ is $\mathbb{G}-T$point of $\hbar_g$, for all $g \in \mathbb{G}$.

Proof :- $\Longrightarrow$ Consider $n = 1$, prove it by induction . Since $\mathbb{G}(\hbar^2(\ell)) \subset \mathbb{G}(\hbar(\ell))$.

Since $\hbar_g(\ell)$ is $\mathbb{G}-SO$, thus there exist $\ell \in \mathcal{L}$ such that

$$\mathbb{G}(\ell) = \{\hbar_g^r : g \in \mathbb{G}, r \geq 0\} \text{ is dense in } \mathcal{L}.$$

This mean $\hbar_g(\ell)$ is $\mathbb{G}-T$ point.

Suppose it is true for $n = r-1$, this means that

$$\overline{\mathbb{G}(\hbar_g^{r-1}(\ell))} = \mathcal{L}.$$

Thus, $\hbar_g^{r-1}(\ell)$ is $\mathbb{G}-T$ point.

Now, for $n = r$, since $\hbar_g^r(\ell) = \hbar_g(\hbar_g^{r-1}(\ell))$, and since $\hbar_g^{r-1}(\ell)$ is $\mathbb{G}-T$ point. Then $\overline{\mathbb{G}(\hbar_g(\hbar_g^{r-1}(\ell)))} = \mathcal{L}$, thus $\hbar_g^r(\ell)$ is $\mathbb{G}-T$ point of $\hbar_g$.

Suppose that for any given $n \in \mathbb{N}$, $\hbar_g^n(\ell)$ $\Longleftarrow$ is a $\mathbb{G}-T$ point of $\hbar_g$, that is mean $\overline{\mathbb{G}(\hbar_g^n(\ell))} = \mathcal{L}$, for some $\ell \in \mathcal{L}$, assume that $n = 1$, then $\hbar_g(\ell)$ is $\mathbb{G}-T$ point, this mean the orbit of $\hbar_g(\ell)$ is dense in $\mathcal{L}$. so by definition 2.6, $\hbar_g(\ell)$ is $\mathbb{G}-SO$ of $\hbar_g$.

■

Proposition 3.5:

Let $\mathcal{L}$ be a $\mathbb{G}-$ space without qu.iso.pts., then $\hbar_g$ is $\mathbb{G}-OT$ if and only if $\hbar_g$ is $\mathbb{G}-SO$.

Proof:- $\Longrightarrow$ Suppose that $\hbar_g$ is $\mathbb{G}-OT$, then there exist $\ell \in \mathcal{L}$such that $\overline{\mathbb{G}(\ell)} = \mathcal{L}$.

this mean the orbit of ℓ is dense, hence ℓ is point transitive. By lemma 3.4, we have that for $n = 1$, $\hbar_g$ is also $\mathbb{G}-T$ point, this mean

$$\overline{\mathbb{G}(\hbar_g(\ell))} = \mathcal{L},$$

so by definition of $\mathbb{G}-SO$ $\hbar_g$ is $\mathbb{G}-SO$, $g \in \mathbb{G}$.

$\Longleftarrow$ Suppose that $\hbar_g$ is $\mathbb{G}-SO$, then there exist $\ell \in \mathcal{L}$ such that $\overline{\mathbb{G}(\hbar_g(\ell))} = \mathcal{L}$,

So, by lemma 3.4, when $n = 1$, $\hbar_g$ is $\mathbb{G}-T$ point, this means that $\hbar_g$ is $\mathbb{G}-OT$, $g \in \mathbb{G}$.

■

***Proposition* 3.6:**

Let $\mathcal{L}$ be a $\mathbb{G}-$ space without qu.iso.pts., if $\hbar_g$ $\mathbb{G}-SO$, then it is $\mathbb{G}-\omega-T$

Proof:- Let $\mathcal{L}$ without qu.iso.pts., then by lemma 3.4, all points in the orbit $\mathbb{G}(\ell)$ are $\mathbb{G}-T$ points of $\hbar_g$ and for any $y \in \mathcal{L}$, any neighborhood $\mathcal{M}$ of $y$in $\mathcal{L}$, any $n \in \mathbb{N}$, there exists $k \geq n$ such that

$$\hbar_{g}^{k}(\ell) \in \mathcal{M}, g \in \mathbb{G}.$$

So $y \in \mathbb{G} - \omega(\ell, \hbar_{g})$. that means

$$\mathbb{G} - \omega(\ell, \hbar_{g}) = \mathcal{L},$$

Therefore, $\hbar_{g}$ is $\mathbb{G} - \omega - T$.

■

Proposition* 3.7*:

Let $\mathcal{L}$ be $\mathbb{G}-$ space without qu.iso.pts.. every $\mathbb{G} - OT$ map $\hbar_{g}$ is open-set$-\mathbb{G} - T$.

Proof :- Let ℓ be a $\mathbb{G} - T$ point of $\hbar_{g}$, so by definition 2.3, $\mathbb{G}(\ell)$ is dense in $\mathcal{L}$ for every $g \in \mathbb{G}$. for any non-empty open sets $\mathcal{M}$ and $\mathcal{N}$ in $\mathcal{L}$, there exists $n \in \mathbb{N}$ such that $\hbar_{g}^{n}(\ell) \in \mathcal{M}$, for all $g \in \mathbb{G}$. by lemma 3.4, $\hbar_{g}^{n+1}(\ell)$ is also $\mathbb{G} - T$ point of $\hbar_{g}$. Thus there exists $k \in \mathbb{N}$ such that

$$\hbar_{g}^{k}\left(\hbar_{g}^{n+1}(\ell)\right) \in \mathcal{N}, g \in \mathbb{G}.$$

implies that,

$$\hbar_{g}^{k+n+1}(\ell) = \hbar_{g}^{n}(\hbar_{g}^{k+1}(\ell)) \in \mathcal{N}, g \in \mathbb{G}.$$

this means that $\hbar_{g}^{k+1}(\mathcal{M}) \cap \mathcal{N} \neq \emptyset$, such that $k + 1 \in \mathbb{N}$. Hence $\hbar_{g}$ is open-set$-\mathbb{G} - T$.

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