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Fractional Calculus and Chaos Synchronization

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Abstract
This paper presents a brief overview of recent developments in chaos synchronization in coupled fractional differential systems, where the original viewpoints are retained. In addition to complete synchronization, several other extended concepts of synchronization, such as projective synchronization, hybrid projective synchronization, function projective synchronization, generalized synchronization and generalized projective synchronization in fractional differential systems, are reviewed.

1. Introduction

Fractional calculus was formulated in 1695, shortly after the development of classical calculus. The earliest systematic studies were attributed to Liouville, Riemann, Leibniz, etc. [**1**,**2**]. An outline of the simple history of fractional calculus can be found in Machado *et al*. [**3**].

For a long time, fractional calculus was regarded as a pure mathematical realm without real applications. But, in recent decades, this has changed. It was found that fractional calculus is useful, even powerful, for modeling viscoelasticity [**4**], electromagnetic waves [**5**], boundary layer effects in ducts [**6**], quantum evolution of complex systems [**7**], distributed-order dynamical systems [**8**] and others. That is, the fractional differential systems are more suitable to describe physical phenomena that have memory and genetic characteristics.

On the other hand, it is known that chaos is ubiquitous in most nonlinear systems. Owing to the various backgrounds of scientific communities, there exist several non-equivalent mathematical definitions of chaos . However, the criterion that the positivity of the largest Lyapunov exponent implies chaos is generally accepted.

The present review has collected most key references on chaos synchronization of fractional differential systems, where the viewpoints of the original contributors are retained. The remainder of the article is organized as follows. In §2, some basic concepts of chaos synchronization of fractional differential systems are introduced. Section 3 reviews the developments in chaos synchronization of coupled fractional-order chaotic systems. The last section concludes this paper.

2. Some basic concepts

Let R, R_+ and Z_+ be the set of real numbers, the set of positive real numbers and the set of positive integer numbers, respectively.

Among several definitions for the fractional derivative, the Caputo derivative and the Riemann–Liouville derivative are most familiar. Engineers like to use the former, whereas physicists and mathematicians often choose the latter. In this paper, the involved fractional derivatives mean the Caputo derivative or the Riemann–Liouville derivative. These two fractional derivatives are not equivalent and have their respective applications.

Definition 2.1

The αth order *Caputo derivative* of a function $f(t)$ is defined by

$$ {}_{C}D^{\alpha}_{0,t}f(t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}f^{(m)}(\tau)\,\mathrm{d}\tau, $$

where $m-1<\alpha\leq m\in Z_+$ and $\Gamma(\cdot)$ is the gamma function.

Definition 2.2

The αth order *Riemann–Liouville derivative* of a function $f(t)$ is defined by

$$ {}_{\mathrm{RL}}D^{\alpha}_{0,t}f(t)=\frac{1}{\Gamma(m-\alpha)}\frac{\mathrm{d}^m}{\mathrm{d}t^m}\int_0^t (t-\tau)^{m-\alpha-1}f(\tau)\,\mathrm{d}\tau, $$

where $m-1\leq\alpha<m\in Z_+$.

Among various kinds of synchronization, CS of two coupled fractional differential systems is the same as that of two coupled conventional differential systems, which are introduced in the appendices A–D. In this article, the fractional partially linear system is used to define CS, PS and HPS.

Definition 2.3

A *fractional partially linear system* is a set of fractional differential equations where the state vector can be decomposed into two parts, (u,z), in which the equation for z is nonlinear in u while that for the fractional derivative of the vector u is linear in z through a matrix M, which depends only on z, in the form of

and
$$ \left.\begin{aligned} \frac{\mathrm{d}^{\alpha}u}{\mathrm{d}t^{\alpha}} &= M(z)\cdot u \\ \frac{\mathrm{d}^{\alpha}z}{\mathrm{d}t^{\alpha}} &= f(u,z), \end{aligned}\right\} $$

where α is the fractional order and $\mathrm{d}^{\alpha}/\mathrm{d}t^{\alpha}$ denotes

either ${}_{C}D^{\alpha}_{0,t}$ or ${}_{\mathrm{RL}}D^{\alpha}_{0,t}$.

Now, some basic definitions about synchronization are given.

Consider two copies of a partially linear system, which are coupled through the variable z in the following manner:

and
$$ \left.\begin{aligned} \frac{\mathrm{d}^{\alpha}u_m}{\mathrm{d}t^{\alpha}} &= M(z)\cdot u_m, \\ \frac{\mathrm{d}^{\alpha}z}{\mathrm{d}t^{\alpha}} &= f(u_m,z) \\ \frac{\mathrm{d}^{\alpha}u_s}{\mathrm{d}t^{\alpha}} &= M(z)\cdot u_s, \end{aligned}\right\} $$

where α is the fractional order, and $u_m\in R^n$ and $u_s\in R^n$ are the state vectors of the drive and response systems, respectively.

Definition 2.4

The two coupled systems in (**2.4**) are said to reach CS if

$$ \lim_{t\to\infty}\|u_m-u_s\|=0, $$

where $\|\cdot\|$ denotes a norm (usually, the Euclidean norm) of a vector.

Here, CS is defined through the fractional partially linear system (**2.4**) just for simplicity and convenience. CS has other coupled forms; see appendices A–D for more details.

Definition 2.5

The two coupled systems (**2.4**) are said to reach PS if, for the initial conditions, there is a constant β such that

$$ \lim_{t\to\infty}\|u_m-\beta u_s\|=0. $$

Definition 2.6

The two coupled systems (**2.4**) are said to reach HPS, if there exist n constants h_i ($1\leq i\leq n$) such that

$$ \lim_{t\to\infty}\|u_m-Hu_s\|=0, $$

where $H=\mathrm{diag}(h_1,h_2,\ldots,h_n)$ is called the scaling matrix and $h_1,h_2,\ldots,h_n$ are the scaling factors.

Definition 2.8

The two coupled systems (**2.8**) are said to reach FPS if there exists a controller $u(x,y)$ such that

$$ \lim_{t\to\infty}\|y-K(x)x\|=0, $$

where $K(x)=\mathrm{diag}(k_1(x),k_2(x),\ldots,k_n(x))$ with $k_i(x)$ being continuous functions, $i=1,2,\ldots,n$.

Next, considering the following two unidirectionally coupled fractional systems:

and
$$\left.\begin{aligned}\frac{\mathrm{d}^p x}{\mathrm{d}t^p} &= f(x)\\ \frac{\mathrm{d}^q y}{\mathrm{d}t^q} &= g(y,u) = g(y,h(x)),\end{aligned}\right\}$$

where $x=(x_1,x_2,\ldots,x_n)^T \in R^n$, $y=(y_1,y_2,\ldots,y_m)^T \in R^m$, $\mathrm{d}^p x/\mathrm{d}t^p=(\mathrm{d}^{p_1}x_1/\mathrm{d}t^{p_1}, \mathrm{d}^{p_2}x_2/\mathrm{d}t^{p_2},\ldots,\mathrm{d}^{p_n}x_n/\mathrm{d}t^{p_n})^T$, $\mathrm{d}^q y/\mathrm{d}t^q=(\mathrm{d}^{q_1}y_1/\mathrm{d}t^{q_1},\mathrm{d}^{q_2}y_2/\mathrm{d}t^{q_2},\ldots,\mathrm{d}^{q_m}y_m/\mathrm{d}t^{q_m})^T$, p_i, $q_j \in R_+$, $p=(p_1,\ldots,p_n)$, $q=(q_1,\ldots,q_m)$, $f: R^n \to R^n$, $g: R^m \times R^k \to R^m$ and $u(t)=(u_1(t),u_2(t),\ldots,u_k(t))^T$ with $u_j(t)=h_j(x(t,x_0))$.

Definition 2.9

The two coupled systems (**2.10**) are said to reach GS if there exist a transformation $H: R^n \to R^m$, a manifold $M=\{(x,y):y=H(x)\}$, and a subset $B=B_x \times B_y \subset R^n \times R^m$ with $M \subset B$, such that, with any initial conditions in B, one has $(x,y) \to M \quad (t \to \infty)$.

Furthermore, consider the following two coupled fractional systems:

and
$$\left.\begin{aligned}\frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha} &= f(x)\\ \frac{\mathrm{d}^\alpha y}{\mathrm{d}t^\alpha} &= g(y,h(x,y)),\end{aligned}\right\}$$

where α is the fractional order, $x \in R^n$, $y \in R^n$, $f: R^n \to R^n$, $h: R^n \times R^n \to R^n$, $g: R^n \times R^n \to R^n$ and $g(x,0) \equiv f(x)$.

Definition 2.10

The two coupled systems (**2.12**) are said to reach GPS if there exists a constant $\sigma \in R-\{0\}$ such that
$$\lim_{t\to\infty} \|x-\sigma y\| = 0.$$

Note that definitions 2.5–2.10 (whose original viewpoints are retained) have some relations but their synchronizations appear in the fractional differential systems with different couplings.

3. Synchronization of fractional chaotic systems

In this section, typical methods for various synchronizations of two coupled fractional chaotic systems are reviewed and discussed.

(a) Complete synchronization

CS can be achieved by means of different coupling schemes. In general, CS can roughly be divided into two categories: unidirectional coupling (drive–response coupling) configuration and bidirectional configuration. In a unidirectional coupling configuration, the evolution of one of the coupled systems is not influenced by the other via coupling. On the contrary, in a bidirectional coupling configuration both systems mutually influence each other . CS is the simplest setting in synchronization of chaotic systems and is easy to apply in practice.

In the following, numerical and analytical methods for CS of the fractional differential systems are introduced.

(i) Numerical methods

There are two popular numerical methods for computing the chaotic attractors of fractional systems and their synchronization diagrams. One is the frequency-domain method and the other is the time-domain method. The former is mainly used to approximate the transfer function $1/s^\alpha$. The latter is used to directly approximate the temporal fractional derivatives. In the study by Li *et al.* , the frequency-domain technique was used to numerically analyse CS of two identical fractional chaotic systems via a one-way coupling configuration (**A1**) (see appendix A), with $k=c\Gamma$, where $c>0$ is the coupling strength and $\Gamma \in R^{n\times n}$ is a constant 0–1 matrix linking the coupling variables. CS of many other fractional chaotic systems via one-way coupling was studied numerically. For example, CS via one-way coupling of two electronic fractional chaotic oscillators in a canonical structure was numerically studied by Gao & Yu , who pointed out that the synchronization rate of a fractional chaotic oscillator was slower than its integer-order counterpart. The one-way coupling technique was also applied to numerically study CS of chaotic fractional Lü systems and of the chaotic fractional Ikeda systems with delays. In the study by Ge & Jhuang , CS of a fractional rotational mechanical system with a centrifugal governor was studied for both autonomous and non-autonomous cases. It was shown that the rotational mechanical system, with its total order less than or more than the number of state variables, exhibited chaos. In addition, it was pointed out that practical chaos synchronization of different fractional systems needs a large coupling strength.

In the study by Tavazoei & Haeri , however, it was pointed out that the time-domain method is more reliable than the frequency-domain method in detecting chaotic attractors of fractional differential systems. One of the most used time-domain methods is the predictor–corrector algorithm . The time-domain method is more flexiable than the frequency-domain method, since approximating the transfer function $1/s^{\alpha}$ is not so convenient if the fractional derivative order α has a large number of digits after the decimal point.

CS of the Chua, Rössler and Chen systems with different fractional orders was investigated numerically by using the predictor–corrector algorithm in the time domain. By selecting proper parameters, numerical results illustrated that synchronization of the fractional Chua, Rössler and Chen systems is slower than that of their respective integer-order systems, where the different fractional orders lie in (0,1).

In addition to the one-way coupling configuration, a control technique was also applied to synchronizing the fractional chaotic systems. For example, the synchronizations of two identical generalized van der Pol systems could be achieved, which was called ‘chaos excited chaos synchronization’ . Chaos synchronization of fractional modified Duffing systems was also studied, and was called ‘parameter excited chaos synchronization’. Moreover, the active sliding mode controller and adaptive proportional–integral–derivative controller were applied to the synchronization of fractional chaotic systems.

(ii) Laplace transform method

The Laplace transform theory was applied by Deng & Li to theoretically study CS of fractional Lü systems by one-way and Pecora–Carroll (PC) coupling configurations (see appendix B). And then the Laplace transform theory was used to theoretically study CS of the Chua systems , the unified chaotic systems and the fractional neuron network systems with time-varying delays . In the study by Li & Deng , the Laplace transform method was applied to investigating CS of the fractional Lorenz systems (x,y,z) in the PC coupling configuration, where (x,z) were driven by y. For coupled fractional Lorenz systems, CS can also be achieved if the driving signal is selected as x , i.e. CS of fractional Lorenz systems can be realized using driving signal x or y, which is in accordance with the case of integer-order Lorenz systems.

Now, the Laplace transform method for synchronization is illustrated by the following examples.

Example 3.1

Consider two identical Chua circuits in a one-way coupling form , in which the drive system is described by

$$\left.\begin{aligned} {}_{C}D^{q_1}_{0,t}x_m(t) &= p_1(y_m - x_m - f(x_m)),\\ {}_{C}D^{q_2}_{0,t}y_m(t) &= x_m - y_m + z_m\\ \text{and}\quad {}_{C}D^{q_3}_{0,t}z_m(t) &= -p_2 y_m,\end{aligned}\right\}$$

and the response system by

$$\left.\begin{aligned} {}_{C}D^{q_1}_{0,t}x_s(t) &= p_1(y_s - x_s - f(x_s)) + k(x_s - x_m),\\ {}_{C}D^{q_2}_{0,t}y_s(t) &= x_s - y_s + z_s\\ \text{and}\quad {}_{C}D^{q_3}_{0,t}z_s(t) &= -p_2 y_s,\end{aligned}\right\}$$

where the fractional orders satisfy $0<q_1,q_2,q_3\leq 1$, k is the coupling

strength, p_1 and p_2 are positive constants, $f(x) = bx + \frac{1}{2}(a-b)(|x+1| - |x-1|)$ with $a<b<0$. The error dynamical system between systems (**3.1**) and (**3.2**) is

$$\left.\begin{aligned} &{}_cD^{q_1}_{0,t}e_1(t) = -p_1e_1 + p_1e_2 - p_1(f(x_s) - f(x_m)) + ke_1,\\ &{}_cD^{q_2}_{0,t}e_2(t) = e_1 - e_2 + e_3\\ \text{and}\quad &{}_cD^{q_3}_{0,t}e_3(t) = -p_2e_2, \end{aligned}\right\}$$

where the error variables $e_1 = x_s - x_m, e_2 = y_s - y_m, e_3 = z_s - z_m$.

Denoting $E_i(s) = \mathscr{L}\{e_i(t)\}, i = 1, 2, 3$, and applying the Laplace transform to both sides of (**3.3**), one obtains

$$\left.\begin{aligned} &s^{q_1}E_1(s) - s^{q_1-1}e_1(0) = -p_1E_1(s) + p_1\mathscr{L}(f(x_s) - f(x_m)) + kE_1(s),\\ &s^{q_2}E_2(s) - s^{q_2-1}e_2(0) = E_1(s) - E_2(s) + E_3(s)\\ \text{and}\quad &s^{q_3}E_3(s) - s^{q_3-1}e_3(0) = -p_2E_2(s). \end{aligned}\right\}$$

With the assumption $|E_3(s)| \le N \in R_+$ and applying the final-value theorem of the Laplace transform , one obtains

$$\lim_{t\to\infty} e_i(t) = \lim_{s\to 0} sE_i(s) = 0, \quad i = 1, 2, 3,$$

which implies that CS between systems (**3.1**) and (**3.2**) is realized. If $q_1=q_2=q_3=1$, system (**3.1**) is the usual Chua system. When the intrinsic parameters are chosen as $p_1=10$, $p_2=14.87$, $a=-1.27$, $b=-0.68$, the usual Chua system has a strange attractor. Similarly, with the same intrinsic parameter values and the order parameters chosen as $q_1=0.92$, $q_2=0.92$, $q_3=0.98$, a chaotic attractor is produced in the uncoupled fractional Chua circuit (**3.1**) (figure 1). With these chosen parameters and $k=16$, the numerical simulation of CS between systems (**3.1**) and (**3.2**) is illustrated in figure 2.

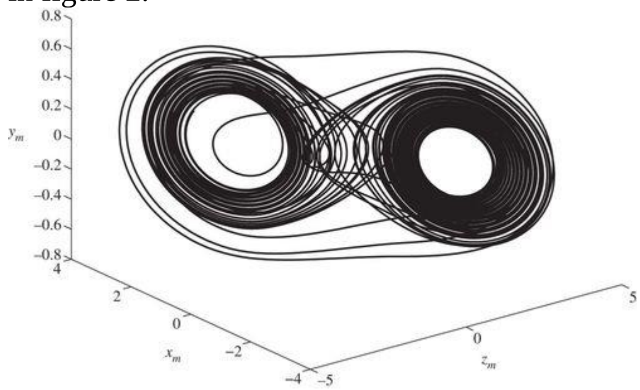

Figure 1.
The fractional Chua circuit in R^3. The diagram shows that the fractional Chua system can also exhibit chaotic behavior, where $p_1=10$, $p_2=14.87$, $a=-1.27$, $b=-0.68$, $q_1=0.92$, $q_2=0.92$, $q_3=0.98$. The time step length is 0.02, the first 100 points are removed.

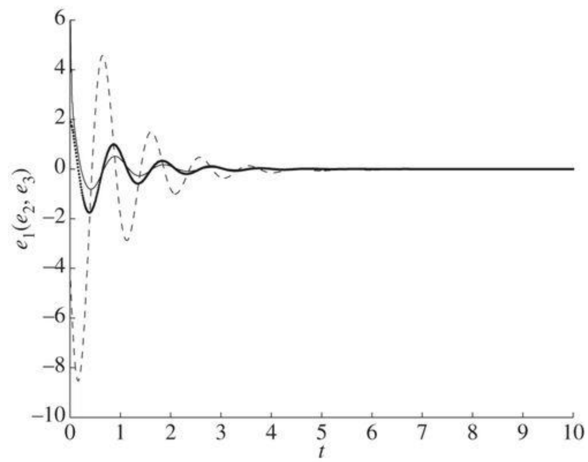

Figure 2.
The evolution diagram of the synchronization errors between (**3.1**) and (**3.2**), which shows that the fractional Chua circuits (**3.1**) and (**3.2**) are asymptotically synchronized. Solid line shows $e_1(t)=x_s-x_m$; dashed line shows $e_2(t)=y_s-y_m$; and dotted line shows $e_3(t)=z_s-z_m$.
Here, $p_1=10$, $p_2=14.87$, $a=-1.27$, $b=-0.68$, $q_1=0.92$, $q_2=0.92$, $q_3=0.98$, $k=16$.

From figure 2, one can see that the fractional Chua circuit (**3.1**) and its slave system (**3.2**) with one-way coupling can also reach CS with the same parameter values as the integer-order forms of (**3.1**) and (**3.2**) by choosing a suitable coupling parameter k.

Remark 3.2

It follows from the above example that the fractional orders chosen are close to 1 in the numerical simulations. In our opinion, according to the conclusion $\lim_{\alpha\to 1^-} {}_C D_{0,t}^{\alpha} x(t) = x^{(1)}(t)$, the fractional system can produce a chaotic attractor similar to its integer-order counterpart with the same parameters.

In the following, this issue is further discussed. For a fractional differential system with a derivative order α lying in (0,1), the smaller the α is taken, the less likely this fractional differential system is to display chaotic behaviour. The reason is possibly that, as α gets smaller and smaller, the stable region becomes larger and larger. For simplicity, take the chaotic fractional Chua circuit [**65**] as an example. When $q_1=q_2=q_3=0.95$, other parameters are the same as those in example 3.1. Figure 3 shows the phase portrait. It can be seen that system (**3.1**) is stable. Then, with $q_1=q_2=q_3=$ 0.96, the system generates a limit cycle, as shown in figure 4. As $q_1=q_2=q_3$ becomes bigger, chaos appears (figure 5) where $q_1=q_2=q_3=0.965$.

When $q_1=q_2=q_3=0.97$ and 0.99, chaotic attractors are found again, and the phase portraits are shown in figures 6 and 7, respectively. With the increase of $q_1=q_2=q_3$, the chaotic attractors are more and more similar to those of the ordinary Chua system. Moreover, $q_1=q_2=q_3=0.96$ is the critical value of transition from stable equilibrium dynamics over self-sustained oscillations to chaos in the fractional Chua system (**3.1**), which is also demonstrated by a one-dimensional bifurcation diagram in figure 8.

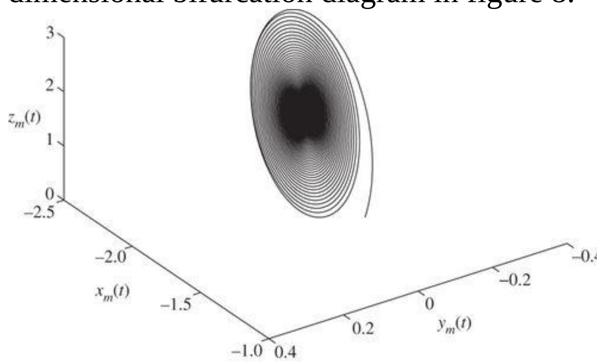

Figure 3. The phase portrait of the fractional Chua system with $q_1=q_2=q_3=0.95$, a stable point.

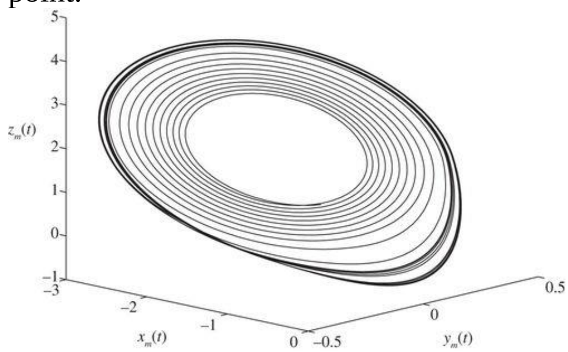

Figure 4. The phase portrait of the fractional Chua system with $q_1=q_2=q_3=0.96$, a stable limit cycle.

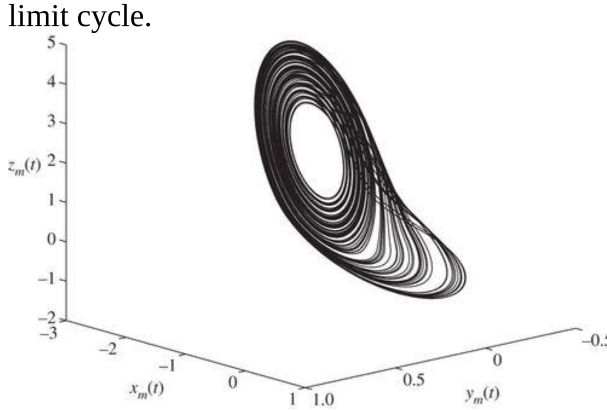

Figure 5. The phase portrait of the fractional Chua system with $q_1=q_2=q_3=0.965$, a chaotic attractor.

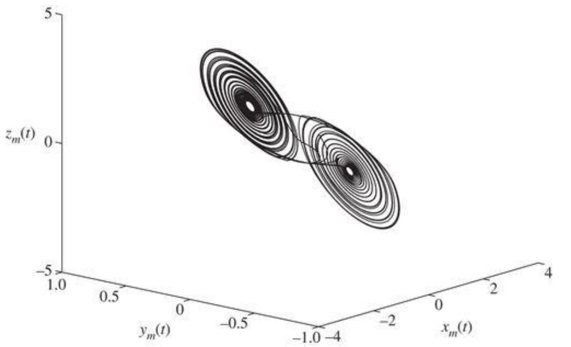

Figure 6. The phase portrait of the fractional Chua system with $q_1=q_2=q_3=0.97$, a chaotic attractor.

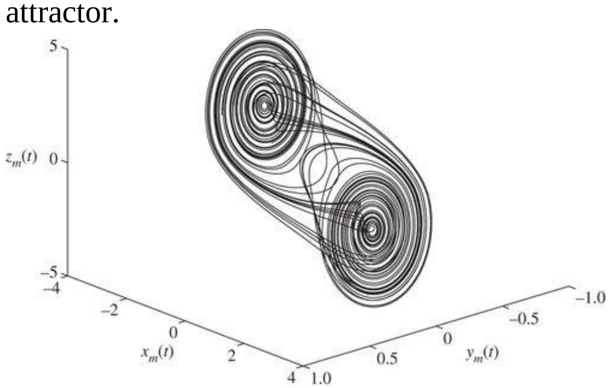

Figure 7. The phase portrait of the fractional Chua system with $q_1=q_2=q_3=0.99$, a chaotic attractor.

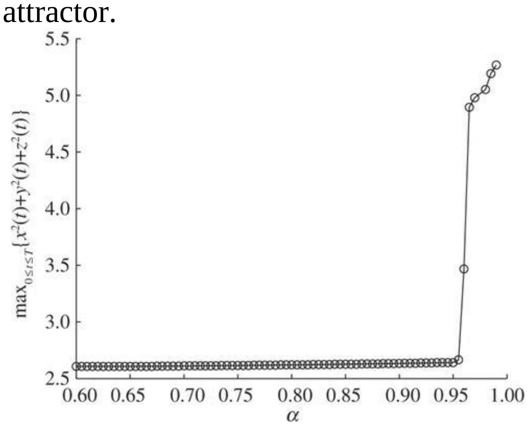

Figure 8. The transition diagram demonstrating the transition from stable equilibrium dynamics over self-sustained oscillations to chaos as the fractal dimension increases in the fractional Chua system (**3.1**). Here, $T=100$, $q_1=q_2=q_3=\alpha$.

Remark 3.3

From example 3.1, the stability analysis of CS between (**3.1**) and (**3.2**) discusses the stability of the zero solution of the error dynamic system of systems (**3.1**) and (**3.2**). Here, the Laplace transform is used. By fixing the parameter values as those in example 3.1 and approximately computing them from the predictor–corrector approach [**71**], one can find that the set of initial conditions leading to synchronization between systems (**3.1**) and (**3.2**) is not arbitrary. Given the drive initial conditions $(x_m(0),y_m(0),z_m(0))=(0.1,-0.2,0.1)$, the set of response initial conditions leading to synchronization between systems (**3.1**) and (**3.2**) lies in

$$W=\{(x_s(0),y_s(0),z_s(0))\mid x_s(0)\ge 1.6 \text{ or } x_s(0)\le -1.3;\ y_s(0)\ge 0.4 \text{ or } y_s(0)\le -0.8;\ z_s(0)\ge 1.6 \text{ or } z_s(0)\le -1.4\},$$

which can be approximately located by numerical calculation.

In the study by Zhu *et al.* [**72**], the Laplace transform method was also applied to investigating CS of the following fractional Chua systems with the coupled matrix (k_1,k_2,k_3), where the drive system is given by

$$\left.\begin{aligned} {}_CD^{q_1}_{0,t}x_m(t)&=p_1(y_m-x_m-f(x_m)),\\ {}_CD^{q_2}_{0,t}y_m(t)&=x_m-y_m+z_m\\ \text{and}\quad {}_CD^{q_3}_{0,t}z_m(t)&=-p_2y_m-p_3z_m,\end{aligned}\right\}$$

and the response system by

$$\left.\begin{aligned} {}_CD^{q_1}_{0,t}x_s(t)&=p_1(y_s-x_s-f(x_s))-k_1(x_s-x_m),\\ {}_CD^{q_2}_{0,t}y_s(t)&=x_s-y_s+z_s-k_2(y_s-y_m)\\ \text{and}\quad {}_CD^{q_3}_{0,t}z_s(t)&=-p_2y_s-p_3z_s-k_3(z_s-z_m),\end{aligned}\right\}$$

in which $f(x)$ is the same as that in example 3.1.

Taking $p_1=10.725$, $p_2=10.593$, $p_3=0.268$, $a=-1.1726$, $b=-0.7872$, $q_1=0.93$, $q_2=0.99$, $q_3=0.92$, the fractional Chua system (**3.6**) also has a chaotic attractor. And, for systems (**3.6**) and (**3.7**), the synchronization thresholds were determined by using bifurcation graphs. Set the coupled matrix (k_1,k_2,k_3) to be $(k,0,0)$. Then, the transition diagrams can be obtained as shown in figure 9.

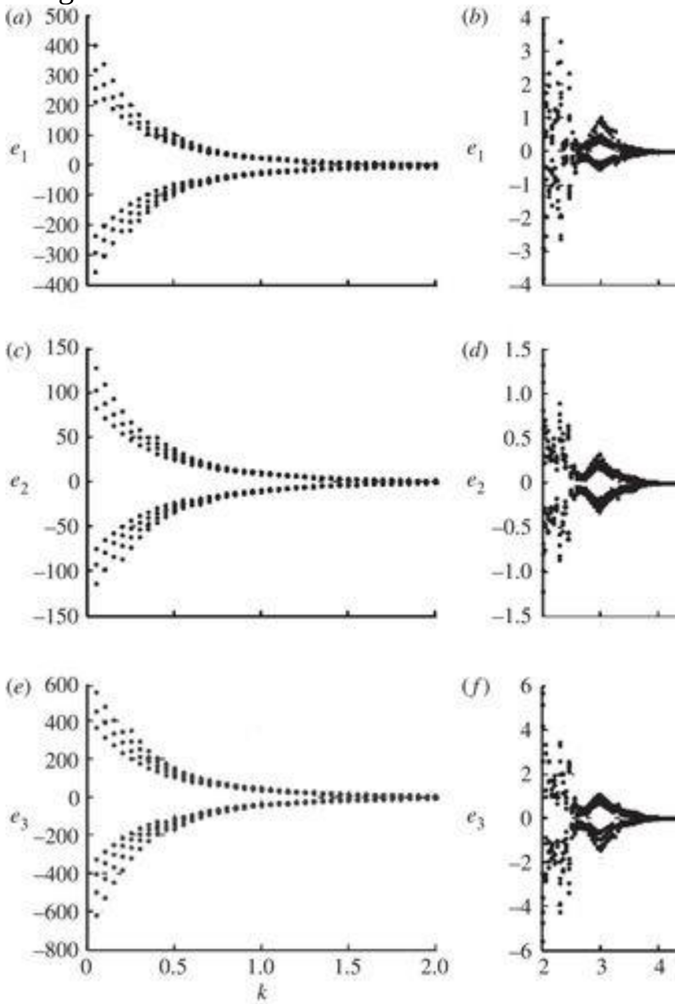

Figure 9.

The error diagrams of the synchronization configuration of the Chua systems (**3.6**) and (**3.7**). (*a*) e_1 versus k, (*b*) e_1 versus k, (*c*) e_2 versus k, (*d*) e_2 versus k, (*e*) e_3 versus k, (*f*) e_3 versus k. Here, $e_1=x_s-x_m$, $e_2=y_s-y_m$, $e_3=z_s-z_m$.

From figure 9, it can be seen that the coupled system (**3.6**) and (**3.7**) with the coupled matrix $(k,0,0)$ is synchronized when the parameter k is greater than 4. Similarly, set the coupled matrix (k_1,k_2,k_3) to be $(k,k,0)$ and $(k,k,0)$ in system (**3.7**), respectively. Then, the synchronization can be realized when the parameter k is greater than approximately 1.0 and 0.5, respectively. Thus, it can be seen that the synchronization rate of the coupled matrix (k,k,k) is the fastest one .

Example 3.4

Consider a PC drive–response configuration with the drive system given by the fractional Lü system (with three state variables denoted by the subscript m) and the response system given by its subsystem containing the (x,z) variables .

The drive system is described by

$$\left.\begin{aligned} {}_C D_{0,t}^{q_1} x_m(t) &= a(y_m - x_m),\\ {}_C D_{0,t}^{q_2} y_m(t) &= -x_m z_m + c y_m\\ \text{and}\quad {}_C D_{0,t}^{q_3} z_m(t) &= x_m y_m - b z_m. \end{aligned}\right\}$$

and the response system by

$$\left.\begin{aligned} {}_C D_{0,t}^{q_1} x_s(t) &= a(y_m - x_s)\\ \text{and}\quad {}_C D_{0,t}^{q_3} z_s(t) &= x_s y_m - b z_s. \end{aligned}\right\}$$

where $0<q_1,q_2,q_3\leq 1$, the response subsystem's variables are denoted by subscript s, and the chaotic signal y_m is used to drive the response subsystem.

Subtracting system (**3.9**) from system (**3.8**) leads to the following error dynamical system:

$$\left.\begin{aligned} {}_C D_{0,t}^{q_1} e_1(t) &= -a e_1\\ \text{and}\quad {}_C D_{0,t}^{q_3} e_3(t) &= y_m e_1 - b e_3. \end{aligned}\right\}$$

where $e_1=x_s-x_m$ and $e_3=z_s-z_m$. Then, applying the Laplace transform to (**3.10**) as in example 3.1, one can achieve CS of systems (**3.8**) and (**3.9**) in the y-drive configuration. This result is illustrated by Deng & Li, with $(a,b,c)=(36,3,20)$ and $q_1=0.985$, $q_2=0.99$, $q_3=0.98$.

When $a=36$, $b=3$, $c=20$, the usual Lü system, i.e. $q_1=q_2=q_3=1$, has a chaotic attractor. Its counterpart also behaves chaotically. Systems (**3.8**) and (**3.9**) can be asymptotically synchronized through a PC drive–response configuration. The diagram of the synchronization errors is provided in Deng & Li .

Remark 3.5

The analysis method in example 3.4 is almost the same as that in example 3.1.

Example 3.6

Consider applying the Laplace transform method to the fractional Chua circuit via the active–passive decomposition (APD) configuration (see appendix C),

$$\left.\begin{aligned} {}_C D^{q_1}_{0,t} x(t) &= p_1(y - x - s(t)), \\ {}_C D^{q_2}_{0,t} y(t) &= x - y + z \\ \text{and} \quad {}_C D^{q_3}_{0,t} z(t) &= -p_2 y, \end{aligned}\right\}$$

driven by

signal $s(t) = f(x) = bx + \frac{1}{2}(a-b)(|x+1| - |x-1|)$ with $a<b<0$, where q_i (i=1,2,3) are positive constants in (0,1].

By the final-value theorem of the Laplace transform, CS between the response system and its replica is implemented. When the coupling configuration is changed to the APD one, the coupled fractional Chua systems can be asymptotically synchronized with the parameter values p_1=10, p_2=14.87, a=−1.27, b=−0.68, q_1=0.92, q_2=0.92, q_3=0.98.

It is worth noting that the PC scheme for synchronization is a special case of the more general APD method. The freedom to choose the driving signal makes the APD scheme flexible in applications. For this reason, the APD scheme is usually combined with the simple one-way method to study CS by using the Laplace transform .

Example 3.7

Consider applying the Laplace transform method to studying synchronization of the fractional Duffing systems by using a combination of the APD method and the one-way coupling method . The drive system is

$$\left.\begin{aligned} {}_C D^{q_1}_{0,t} x_m(t) &= y_m \\ \text{and} \quad {}_C D^{q_2}_{0,t} y_m(t) &= -\tfrac{1}{25} y_m + \tfrac{1}{5} x_m - \tfrac{8}{15} s(t) + \tfrac{2}{5}\cos(0.2t), \end{aligned}\right\}$$

and the response system is

$$\left.\begin{aligned} {}_C D^{q_1}_{0,t} x_s(t) &= y_s + u(x_s - x_m) \\ \text{and} \quad {}_C D^{q_2}_{0,t} y_s(t) &= -\tfrac{1}{25} y_s + \tfrac{1}{5} x_s - \tfrac{8}{15} s(t) + \tfrac{2}{5}\cos(0.2t), \end{aligned}\right\}$$

where $0<q_1,q_2\le 1$, u is a control parameter, and $s(t) = x_m^3$ is regarded as the driving signal.

If u=0, then this drive–response configuration corresponds to the APD method. If $s(t) = x_m^3$ in the drive system and $s(t) = x_s^3$ in the response system, then it corresponds to the one-way coupling method. Applying the Laplace transform to the corresponding final-value theorem, the CS state can be realized as long as $u \neq -5$. By comparing the diagrams of the synchronization errors, it is found that this synchronization method is more effective for the Duffing system, since reaching synchronization takes longer than using only the APD scheme .

Apart from the aforementioned unidirectional coupling configuration, there is a more effective bidirectional coupling method (see appendix D) for CS of fractional chaotic systems. By applying the bidirectional coupling scheme to a pair of coupled fractional Rössler systems,

$$\left.\begin{aligned} {}_C D^{q_1}_{0,t} x_m &= -y_m - z_m + c_1(x_s - x_m) \\ {}_C D^{q_2}_{0,t} y_m &= x_m + a y_m + c_2(y_s - y_m) \\ {}_C D^{q_3}_{0,t} z_m &= 0.2 + z_m(x_m - 10) + c_3(z_s - z_m) \end{aligned}\right\} \text{drive}$$

and

$$\left.\begin{aligned} {}_C D^{q_1}_{0,t} x_s &= -y_s - z_s + c_1(x_m - x_s) \\ {}_C D^{q_2}_{0,t} y_s &= x_s + a y_s + c_2(y_m - y_s) \\ {}_C D^{q_3}_{0,t} z_s &= 0.2 + z_s(x_s - 10) + c_3(z_m - z_s), \end{aligned}\right\} \text{response}$$

one has the following error dynamical system:

$$\left.\begin{aligned} {}_C D^{q_1}_{0,t} e_1 &= -e_2 - e_3 - 2c_1 \cdot e_1, \\ {}_C D^{q_2}_{0,t} e_2 &= e_1 + a e_2 - 2c_2 \cdot e_2 \\ \text{and} \quad {}_C D^{q_3}_{0,t} e_3 &= z_s e_1 + x_m e_3 - 10 e_3 - 2c_3 \cdot e_3, \end{aligned}\right\}$$

where $0<q_1,q_2,q_3\le 1$, $e_1=x_s-x_m$, $e_2=y_s-y_m$ and $e_3=z_s-z_m$. By using the Laplace transform and the final-value theorem, CS between systems (**3.14**) and (**3.15**) can be achieved under some prior assumptions. Select a=0.4 and $q_1=q_2=q_3$=0.9, so as to produce chaotic dynamics in the uncoupled fractional Rössler system. With these parameters and c_1=0.8, $c_2=c_3$=0.6, all the synchronization errors e_i (i=1,2,3) soon converge to zero. The synchronization error evolution of the bidirectional coupling method is shown

in figure 10.

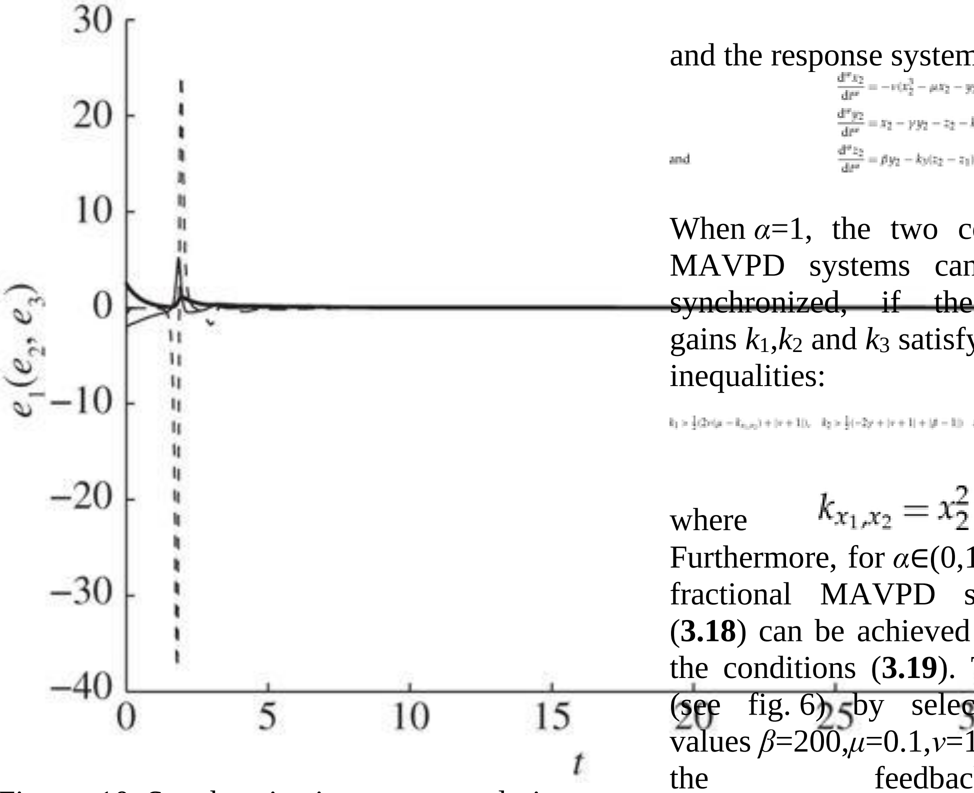

Figure 10. Synchronization error evolution of the drive–response systems (**3.14**) and (**3.15**) with the bidirectional coupling method, where the phase curves of synchronization errors show that the synchronized chaotic state is realized, where c_1=0.8, c_2=c_3=0.6 and q_1=q_2=q_3=0.9. Here, solid line shows $e_1(t)=x_s-x_m$; dotted line shows $e_2(t)=y_s-y_m$; and dashed line shows $e_3(t)=z_s-z_m$.

(iii) Stability analysis

In this section, the stability theory of fractional systems is applied to studying CS of fractional chaotic systems with various kinds of couplings. It is well known that the stability region of the fractional case is greater than the stability region of the corresponding integer-order case if the fractional order lies in (0,1). Based on this fact, CS of fractional modified autonomous Van der Pol–Duffing (MAVPD) circuits was studied by a one-way coupling scheme as follows.

The drive system is

$$\frac{d^\alpha x_1}{dt^\alpha} = -\nu(x_1^3 - \mu x_1 - y_1), \quad \frac{d^\alpha y_1}{dt^\alpha} = x_1 - \gamma y_1 - z_1 \quad \text{and} \quad \frac{d^\alpha z_1}{dt^\alpha} = \beta y_1,$$

and the response system is

$$\left.\begin{aligned} \frac{d^\alpha x_2}{dt^\alpha} &= -\nu(x_2^3 - \mu x_2 - y_2) - k_1(x_2 - x_1), \\ \frac{d^\alpha y_2}{dt^\alpha} &= x_2 - \gamma y_2 - z_2 - k_2(y_2 - y_1) \\ \text{and} \quad \frac{d^\alpha z_2}{dt^\alpha} &= \beta y_2 - k_3(z_2 - z_1). \end{aligned}\right\}$$

When α=1, the two coupled integer-order MAVPD systems can be asymptotically synchronized, if the feedback control gains k_1,k_2 and k_3 satisfy the following inequalities:

$$k_1 > \tfrac{1}{2}(2\nu(\mu - k_{x_1,x_2}) + |\nu + 1|), \quad k_2 > \tfrac{1}{2}(-2\gamma + |\nu + 1| + |\beta - 1|) \quad \text{and} \quad k_3 > \tfrac{1}{2}(|\beta - 1|),$$

where $k_{x_1,x_2} = x_2^2 + x_1 x_2 + x_2^2 \geq 0$. Furthermore, for $\alpha \in (0,1]$, CS of the coupled fractional MAVPD systems (**3.17**) and (**3.18**) can be achieved if k_i (i=1,2,3) satisfy the conditions (**3.19**). This can be verified (see fig. 6) by selecting the parameter values β=200,μ=0.1,ν=100,γ=1.6,α=0.98 and the feedback control gains k_1=280,k_2=250,k_3=100, which satisfy the inequalities (**3.19**).

In addition, one can apply the stability theory to studying CS of fractional chaotic systems by one-way coupling . Especially, based on the stability theory of delayed fractional differential systems, CS of delayed fractional chaotic systems by one-way coupling was investigated by Deng *et al.*, who simulated CS of the coupled Duffing oscillators.

Next, the stability theory of fractional differential systems is employed to investigate CS of fractional chaotic systems with the PC drive–response configuration. Consider the PC drive–response configuration with the drive system given by the fractional Chen system (with subscript m)

$$\left.\begin{aligned} {}_C D_{0,t}^{\alpha} x_m &= a(y_m - x_m), \\ {}_C D_{0,t}^{\alpha} y_m &= (c - a)x_m - x_m z_m + c y_m \\ \text{and} \quad {}_C D_{0,t}^{\alpha} z_m &= x_m y_m - b z_m. \end{aligned}\right\}$$

and the response system chosen as the subsystem of (x,z)

and

$$\left.\begin{aligned} {}_{C}D_{0,t}^{\alpha_2} x_s &= a(y_m - x_s) \\ {}_{C}D_{0,t}^{\alpha_3} z_s &= x_s y_m - b z_s. \end{aligned}\right\}$$

For the error dynamical system of systems (**3.20**) and (**3.21**), by applying the stability theorem of multi-rational-order fractional differential systems , CS is achieved for the parameters $(a,b,c)=(35,3,28)$, $(\alpha_1,\alpha_2,\alpha_3)=(0.9,0.95,0.95)$. For the fractional Lorenz system, several PC drive–response configurations were studied with the drive system given by the same order fractional Lorenz system and the response system given by its subsystems containing one state variable and two state variables. The stability theorem of fractional differential systems was applied to discuss all possible drive–response subsystems, which can divide the Lorenz system. With the drive system containing one state variable, only two choices can induce CS, which agrees with the integer-order Lorenz system case. Yet, all possible choices can induce the appearance of CS when the drive system contains two state variables .

Conclusion

This paper presents an overview of chaos synchronization of coupled fractional differential systems. A list of coupling schemes is presented, including one-way coupling, PC coupling, APD coupling, bidirectional coupling and other unidirectional coupling configurations. Also, several extended concepts of synchronization are introduced, namely, PS, HPS, FPS, GS and GPS. Corresponding to different kinds of synchronization schemes, various analysis methods are presented and discussed.

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