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Double Stage Bayes Pre- Test shrinkage estimation for exponential distribution

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Abstract

This paper deal with double bayes estimation of exponential distribution when prior information exists as an initial value using progressive censored sample type II by suggested and study properties of two pre-test Bays shrinkage estimators. The Equations bias, bias ratio, and mean square error, excepted sample size are derived and studied numerical, the results show that the proposed estimators have higher relative efficiency compared with the classical Bayesian estimator.

Keywords: Exponential Distribution; Shrinkage Estimation; Bayes Estimator; Mean Square Error; Relative efficiency and Progressive Type II.

1. INTRODUCTION

Exponential distribution is used in a variety of fields, including problems related to time measurement. In the field sciences, the exponential distribution applications included, failure rate of electronic device, predicting how long a machine will run before unplanned downtime and how long until the next email comes through your Inbox at work, etc.

A continuous random variable X has exponential distribution if the following probability density function (p.d.f.) as

$$f(x\,;\theta\,) = \theta \exp(\,-x\theta) \qquad ; \quad x > 0\,, \quad \theta > 0\,. \qquad (1)$$

and the cumulative density function (c.d.f) as:

$$f(x\,;\theta\,) = 1 - \exp(\,-x\theta) \;\; ; \quad x > 0,\ \theta > 0\,. \qquad (2)$$

where θ is the scale parameter.

Parameter estimation is essential topic in statistical inference because it assists to determine the complete description of life distributions that could fit with the data. When previously knowledge about the unknown parameters is rare value, we use conventional estimation approximation as maximum likelihood method, such as moments and uniformly minimal variance unbiased estimator, and so on. When prior knowledge exists, it must be included in the estimation. The Bayes estimation method is useful in this case, in particular if that previously knowledge about the unknown parameter θ as a prior distribution. When the prior information is only available as initial value, i.e., θ_0 of θ then we must use it, [Thompson 1968] one of first researcher refer to for use initial value in estimation procedure by made liner combination between the classical estimator θˆ and a prior information θ_0 .

In case, if one is not sure whether the true value of unknown parameter θ is close to an initial value θ_0 or not, the procedure suggested is to make preliminary test for the hypothesis H_o: θ =θ_0 against H_1: θ ≠ θ_0 If the hypothesis H_o: θ =θ_0accepted then we consider the shrinkage estimator as

$$K\hat{\theta}(\,1 - k)\theta_0$$

Where θˆis estimator based the random sample obtained through well-known estimation procedure and k shrinkage factor such that 0≤ k ≤1. Otherwise, the estimator is the usual estimator θˆ. Thus, the single stage shrunken estimator is given as

$$\tilde{\theta} = \begin{cases} k\hat{\theta} + (1-k)\theta_0 & if\ H_0\ is\ accepted \\ \hat{\theta} & e.w. \end{cases} \quad (3)$$

In many situations the sample units is costly or difficult to obtain it therefore must use inexpensive procedure estimation, the double stage shrinkage estimation is good procedure to achieve this goal.[Katti 1962] first author refer to double stage shrinkage estimation used to estimate the mean of normal distribution , when the variance σ2 is known, If the mean of the first sample belongs to area R then it will be used as estimate of μ , otherwise the second sample will be drown then pooled mean will be constructed $\frac{m_1\bar{x}_1+m_2\bar{x}_2}{m_1+m_2}$ to calculate μ.[Shah 1964] applied the method of researcher [Katti 1962] on variance of normal distribution when the mean is unknown, according to [Arnold and Al-Bayyati 1970,1972] attempt to weight θ_0 and θ by a constant k , if θˆ ∈R the estimate $k\,\hat{\theta}_1 + (1-K)\theta_0$ where k is a constant specified by the experimenter according to his belief in θ_0 .

Sometimes, it's difficult take an complete sample therefore need to obtain an censored sample, the researcher or experimenter needs to preserve the ages of the units for use in posterior experiment and wants to raise these units before failure time (R) by using one of the most important censoring sample is progressive censored sample to reducing the total time and the associated cost of testing.

Type-II progressive censoring is one of the censoring methods frequently used in clinical studies, reliability and life testing, quality control of products and industrial experiments. Progressive type II censored sample can be described as follow (see Balakrishnan and Aggarwala (2000)). After observing first failure, R1 units are randomly selected and removed; after observing second failure, R2 units are randomly selected and removed; and likewise when the i-th failure is observed Ri units are randomly selected and removed; i=3, 4,… ,m. The experiment terminates when the m-th failure is observed and the remaining $R_m = n - m - \sum_{i=1}^{m-1} R_i$

The aim of this paper, it's suggest and study properties double stage Bayes shrinkage estimation exponential distribution when the prior information available as initial value.

2. proposed estimators

Assume X1:1:1 , X2:2:2 ,…, Xm:m:m be a random progressive size m from the exponential distribution , The Joint function of the progressive censored sample X1:1:1 , X2:2:2 ,…, Xm:m:m and expressed as

$$f(x_{1:1:1}, \ldots, x_{m,m,m}) = C \prod_{i=1}^{m} f(x_{i:i:i})\,[1 - f(x_{i:i:i})]^{R_i} \quad (4)$$

Where

$$C = n(n - R_1 - 1).(n - R_1 - R_2 - 2)\ldots\ldots(n - \sum_{i=1}^{m-1} R_i - m + 1)$$

[Robert, Chopin 2009] suggested Jeffreys's Theory of Probability revisited .

$g(\theta)\ \alpha\ \sqrt{I(\theta)}$, $where\ I(\theta)$ is Fisher information.

$$H(x_1, x_2, \ldots, x_m, \theta) = \prod_{i=1}^{m} f(x_i\theta)\, g(\theta) \quad (5)$$

In this research, Jeffery's non-informative prior is used as a prior distribution, i.e.., $g(\theta) = \frac{1}{\theta^c}$, $c > 0$. The posterior distribution of θ can be obtained by combining the prior distribution and the likelihood Along with Bayes theorem.

$$\pi(\theta|x_1, x_2, \ldots x_m) = \frac{H(x_1, x_2, \ldots x_m, \theta)}{\int_0^\infty H(x_1, x_2, \ldots x_m, \theta)d\theta} \quad (6)$$

One can easy to show the posterior as follow

$$\pi(\theta;\ x_1, x_2, \dots x_m) = \frac{y^{m-C+1}\theta^{m-c}e^{-\theta y_i}}{\Gamma(m-c+1)} \quad (7)$$

By using the squared error loss function the Bayes estimator is equal to the mean of posterior distribution that is mean i.e.

$$\tilde{\theta}_B = E(\theta) = \int_0^\infty \theta\pi(\theta|x)d\theta \quad (8)$$

$$= \int_0^\infty \theta\pi\,(\theta|x_1, \dots, x_m)\, d\theta$$

$$= \int_0^\infty \theta\ \frac{y^{m-c+1}\ \theta^{m-C} exp(-y\theta)}{\Gamma(m-c+1)}$$

$$= \frac{y^{m-c+1}}{\Gamma(m-c+1)} \int_0^\infty \theta^{m-c+1}\ exp(-y\theta)d\theta$$

by using the transformation

$q=y\,\theta$, we have $\theta = \frac{q}{y}$

then

$$\hat{\theta}_B = \frac{y^{m-c+1}}{\Gamma(m-c+1)} \int_0^\infty (\frac{q}{y})^{m-c+1}\ exp\left(-y\ \frac{q}{y}\right) \frac{1}{y}\, dq$$

$$= \frac{y^{m-c+1}}{\Gamma(m-c+1)\, y^{m-c+2}} \int_0^\infty q^{\ m-c+1}\ exp(-q)\ \frac{1}{y}\, dq$$

$$= \frac{\Gamma(m-c+2)}{\Gamma(m-c+1)\, y}$$

Thus

$$\hat{\theta}_B = \frac{m-c+1}{y} \quad (9)$$

Assume that X has exponential distribution and $\sum_{i=1}^m x_i$ has an gamma distribution with the parameter (m, θ) , and $\frac{1}{\hat{\theta}_B}$ has a Gamma distribution with the parameter (m, (m-c+1) θ). Thus ,

$$f(y\ ,\ \theta) = \frac{\theta^m\ y^{m-1}e^{-\theta y}}{(m-1)!} \quad (10)$$

Then y=($\hat{\theta}$) , |y| =| $\frac{dy}{d\,\hat{\theta}_B}$ | = | $-\ \frac{m-c+1}{\hat{\theta}_B^2}$ |

$$|J| = \frac{m-c+1}{\hat{\theta}_B^2}$$

Therefore, above a transformation one can show the $\hat{\theta}_B$ has density function as

$$f\left(\hat{\theta}_B;\ \theta\right) = \begin{cases} \dfrac{((m-c+1)\theta)^m \left(\frac{1}{(\hat{\theta}_B)}\right)^{m+1} \exp(\frac{(m-c+1)\theta}{\hat{\theta}_B})}{(m-1)!} & \hat{\theta}_B \\ 0 & e. \end{cases} \quad , \quad (11)$$

Depending on above Bayes estimator and the density function of Bayes estimator and by using (Arnold and Al-Bayyati) (1970-1972) procedure to suggest the following Bayes pre-test double stage shrinkage estimator of exponential distribution and using progressive censored sample as following

The first suggested double stage shrinkage estimator ($\tilde{\theta}_{BSD1}$) is defined as

$$\tilde{\theta}_{BSD_1} = \begin{cases} k_1\hat{\theta}_{BS_1} + (1-k_1)\theta_0 & if\ H_0\ is\ accepted \\ \dfrac{m_1\,\hat{\theta}_{BS_1} + m_2\hat{\theta}_{BS_2}}{m_1 + m_2} & e.w. \end{cases}$$

(12)

Where K_1 the shrinkage factor is constants such that ($0 < K_1 < 1$) while R is acceptant region of test the hypothesis $H_0: \theta = \theta_0$ against $H_1: \theta \neq \theta_0$ for level of significance α.

The second suggested double stage shrinkage estimator $\tilde{\theta}_{\mathrm{BSD2}}$

$$\tilde{\theta}_{\mathrm{BSD}_2} = \begin{cases} k_2 \hat{\theta}_{\mathrm{BS}_1} + (1 - k_2)\theta_0 & if\ H_0\ \ is\ accepted \\ \frac{m_1 \hat{\theta}_{\mathrm{BS}_1} + m_2 \hat{\theta}_{\mathrm{BS2}}}{m_1 + m_2} & e.w. \end{cases} \quad (13)$$

Where K_2 the shrinkage factor suggested based on accepted of the hypothesis $H_0: \theta = \theta_0$ and against $H_1: \theta \neq \theta_0$, i.e.

$$r_1 < \frac{2(m-c+1)\theta_0}{\hat{\theta}_{BS}} < r_2$$

Then

$$0 < \frac{2(m-c+1)\theta_0}{\hat{\theta}_{BS}} - r_1 < r_2 - r_1 < r_2$$

Therefore

$$0 < \frac{2(m-c+1)\theta_0}{\hat{\theta}_{BS}\ r_2} - \frac{r_1}{r_2} < 1$$

$$K_2 = \frac{2(m-c+1)\theta_0}{\hat{\theta}_{BS}\ r_2} - \frac{r_1}{r_2} \quad (14)$$

Bayes and mean square error of proposed estimator of $\tilde{\theta}_{BSD_1}$

Now, the Bias estimator and mean square error equation derived as followed

The suggested estimated Bias $\tilde{\theta}_{BSD1_1}$ defined as :

Bias ($\tilde{\theta}_{BSD_1}/\theta$) =E ($\tilde{\theta}_{BSD_1} - \theta$)

$$= \int_0^\infty \int_{\bar{R}} [\ K_1 \hat{\theta}_{\mathrm{BS1}} + (1 - K_1)\theta_0]\ f_1\left(\hat{\theta}_{\mathrm{BS1}}\right) f_2(\hat{\theta}_{\mathrm{BS2}})\ d\hat{\theta}_{\mathrm{BS1}}\ d\hat{\theta}_{\mathrm{BS2}} + \int_0^\infty \int_{\bar{R}} [\ \frac{m_1 \hat{\theta}_{\mathrm{BS1}_1} + m_2 \hat{\theta}_{\mathrm{BS2}}}{m_1 + m_2}]\ f_1\left(\hat{\theta}_{\mathrm{BS1}}\right) f_2\left(\hat{\theta}_{\mathrm{BS2}}\right) d\hat{\theta}_{\mathrm{BS1}}\ d\hat{\theta}_{\mathrm{BS2}} - \theta$$

As utilization the same transformation :

$$\mathrm{M} = \frac{(m_1 - c + 1)\theta}{\hat{\theta}_{\mathrm{BS1}}} \Rightarrow \hat{\theta}_{\mathrm{BS1}} = \frac{(m_1 - c + 1)\theta}{\mathrm{M}} \Rightarrow$$

$$d\ \hat{\theta}_{\mathrm{BS1}} = \frac{(m_1 - c + 1)\theta}{\mathrm{M}^2}$$

$$N = \frac{(m_2 - c + 1)\theta}{\hat{\theta}_{\mathrm{BS2}}} \Rightarrow \hat{\theta}_{\mathrm{BS2}} = \frac{(m_2 - c + 1)\theta}{\mathrm{M}} \Rightarrow$$

$$d\ \hat{\theta}_{\mathrm{BS2}} = \frac{(m_2 - c + 1)\theta}{\mathrm{N}^2}$$

Such that R* equal to R after above the transformation i.e. R*= ($\acute{r}_1, \acute{r}_2$) $and\ \lambda = \frac{\theta_0}{\theta}$

After simplifying

$$\text{Bias}\ \tilde{\theta}_{BSD_1} = \Big[\frac{K_1(m_1 - c + 1)\theta}{(m_1 - 1)!}\{\ I(\acute{r}_2, m_1 - 1) - I(\acute{r}_1, m_1 - 1)\} + \lambda\theta(1 - K_{\ 1})\{\ I(\acute{r}_2, m_1) - I(\acute{r}_1, m_1)\} + \theta[\frac{m1(m1 - c + 1)}{(m_1 + n_2)(m_1 - 1)} - \frac{m1(m1 - c + 1)}{(m_1 + m_2)(m_1 - 1)}]\{\ I(\acute{r}_2, m_1 - 1) - I(\acute{r}_1, m_1 - 1)\} + \theta[\frac{m_2(m_2 - c + 1)}{(m_1 + m_2)(m_2 - 1)} - \frac{m_2(m1 - c + 1)}{(m_1 + m_2)(m_2 - 1)}]\{\ I(\acute{r}_2, m_1) - I(\acute{r}_1, m_1)\} - \theta\Big] \quad (15)$$

Now , the mean square error suggested

estimated ($\tilde{\theta}_{BSD1}$)

$$MSE\ (\tilde{\theta}_{BSD1}) = E(\tilde{\theta}_{BSD1} - \theta)^2$$

$$= \int_0^\infty \int_{\bar{R}} [\ K_1\, \hat{\theta}_{BS1} + (1 - K_1)\theta_0]^2\, f_1\left(\hat{\theta}_{BS1}\right) f_2(\hat{\theta}_{BS2})\, d\hat{\theta}_{BS^1}\, d\hat{\theta}_{BS2}$$

$$+ \int_0^\infty \int_{\bar{R}} \left[\frac{m_1\hat{\theta}_{BS1_1}+m_2\hat{\theta}_{BS2}}{m_1+m_2} - \theta\right]^2 f_1\left(\hat{\theta}_{BS1}\right) f_2\left(\hat{\theta}_{BS2}\right) d\hat{\theta}_{BS1}\, d\hat{\theta}_{BS2}$$

As utilization the same transformation :

$$\mathrm{M} = \frac{(m_1-c+1)\theta}{\hat{\theta}_{\mathrm{BS1}}} \Rightarrow \hat{\theta}_{\mathrm{BS1}} = \frac{(m_1-c+1)\theta}{\mathrm{M}} \Rightarrow$$

$$d\,\hat{\theta}_{\mathrm{BS1}} = \frac{(m_1-c+1)\theta}{\mathrm{M}^2}$$

$$N = \frac{(m_2-c+1)\theta}{\hat{\theta}_{\mathrm{BS2}}} \Rightarrow \hat{\theta}_{\mathrm{BS2}} = \frac{(m_2-c+1)\theta}{\mathrm{M}} \Rightarrow$$

$$d\,\hat{\theta}_{\mathrm{BS2}} = \frac{(m_2-c+1)\theta}{\mathrm{N}^2}$$

And same straight for

$$MSE\ (\tilde{\theta}_{BSD1}) =$$

$$\theta^2\Big[k_1^2\left[\frac{(m_1-c+1)^2}{(m_1-1)(m_1-2)}\right]\{\, I(\acute{r}_2, m_1-2) - I(\acute{r}_1, m_1-2)\} - \frac{2\lambda(m_1-c+1)}{(m^1-1)}\{\, I(\acute{r}_2, m_1-1) - I(\acute{r}_1, m_1-1)\} + \lambda^2\{\, I(\acute{r}_2, m_1) - I(\acute{r}_1, m_1)\} - 2K_1(1-\lambda)\frac{(m_1-c+1)}{(m_1-1)}\{\, I(\acute{r}_2, m_1-1) - I(\acute{r}_1, m_1-1)\} - \lambda\{\, I(\acute{r}_2, m_1) - I(\acute{r}_1, m_1)\}\Big] + (1-\lambda)^2\{\, I(\acute{r}_2, m_1) - I(\acute{r}_1, m_1)\} +$$

$$\frac{m_1^2(m_1-c+1)^2}{(m_1-m_2)(m_1-1)(m_1-2)} - \frac{2\,m_1^2(m_1-c+1)}{(m_1+m_2)^2(m_1-1)} + \frac{m_1^2}{(m_1+m_2)^2} + \frac{2m_1m_2(2-c)^2}{(m_1+m_2)^2(m_1-1)(m_2-1)} + \frac{m_2^2(m_2-c+1)^2}{(m_1+m_2)^2(m_2-1)(m_2-2)} - \frac{2m_2^2(m_2-c+1)}{(m_1+m_2)^2(m_2-1)} + \frac{2m_2^2}{(m_1+m_2)^2} \quad (16)$$

Bayes and mean square error of proposed estimator of $\tilde{\boldsymbol{\theta}}_{BSD_2}$

The suggested estimated Bias $\tilde{\theta}_{BSD_1}$ defined as :

Bias ($\tilde{\theta}_{BSD_2}/\theta$) =E $\left(\tilde{\theta}_{BSD_2} - \theta\right)$

$$= \int_0^\infty \int_{\bar{R}} [\ K_2\, \hat{\theta}_{\mathrm{BS1}} + (1 - K_2)\theta_0]\, f_1\left(\hat{\theta}_{\mathrm{BS1}}\right) f_2(\hat{\theta}_{\mathrm{BS2}})\, d\hat{\theta}_{\mathrm{BS}^1}\, d\hat{\theta}_{\mathrm{BS2}} +$$

$$\int_0^\infty \int_{\bar{R}} \left[\frac{m_1\hat{\theta}_{\mathrm{BS1}_1}+m_2\hat{\theta}_{\mathrm{BS2}}}{m_1+m_2}\right] f_1\left(\hat{\theta}_{\mathrm{BS1}}\right) f_2\left(\hat{\theta}_{\mathrm{BS2}}\right) d\hat{\theta}_{\mathrm{BS1}}\, d\hat{\theta}_{\mathrm{BS2}} - \theta$$

As utilization the same transformation :

$$\mathrm{M} = \frac{(m_1-c+1)\theta}{\hat{\theta}_{\mathrm{BS1}}} \Rightarrow \hat{\theta}_{\mathrm{BS1}} = \frac{(m_1-c+1)\theta}{\mathrm{M}} \Rightarrow$$

$$d\,\hat{\theta}_{\mathrm{BS1}} = \frac{(m_1-c+1)\theta}{\mathrm{M}^2}$$

$$N = \frac{(m_2-c+1)\theta}{\hat{\theta}_{\mathrm{BS2}}} \Rightarrow \hat{\theta}_{\mathrm{BS2}} = \frac{(m_2-c+1)\theta}{\mathrm{M}} \Rightarrow$$

$$d\,\hat{\theta}_{\mathrm{BS2}} = \frac{(m_2-c+1)\theta}{\mathrm{N}^2}$$

$$Bias\ (\tilde{\theta}_{BSD_2}/\theta) =$$

$$\left[\frac{2(m_1-c+1)}{r_2(m_1-1)!}\right. \int_{R^*} m^{m-1}\ e^{-m} dm - \frac{r_1\lambda(m_1-c+1)}{r^2(m_1-1)!} \int_{R^*} m^{m-2}\ e^{-m} dm -$$

$\frac{\lambda(1+r_1\backslash r_2)}{(m_1-1)!} \int_{R^*} m^{m-2}\ e^{-m} dm\ -$

$2\ \frac{\lambda^2}{r_2} \int_{R^*} m^{m-2}\ e^{-m} dm +$

$\frac{m_1(m_1-c+1)}{(m_1+m_2)(m_1-1)} - \frac{m_1(m_1-c+1)}{(m_1+m_2)(m_1-1)}\Big]\ \ \{I\ (\acute{r}_2, m_1 -$

$1) - I\ (\acute{r}_1, m_1-1)\} + \frac{m_2(m_2-c+1)}{(m_1+m_2)(m_2-1)}\ -$

$\frac{m_2(m_2-c+1)}{(m_1+m_2)(m_2-1)} + \{I\ (\acute{r}_2, m_1) - I\ (\acute{r}_1, m_1)\} - 1$

(17)

Now , the mean square error suggested estimated ($\tilde{\theta}_{BSD2}$)

$MSE\ (\tilde{\theta}_{BSD2}) = E(\tilde{\theta}_{BSD2} - \theta)^2$

$= \int_0^\infty \int_{\bar{R}} [\ K_2\, \hat{\theta}_{BS1} + (\ 1 -$

$K_2)\theta_0]^2\, f_1\left(\hat{\theta}_{BS1}\right) f_{\,2}(\hat{\theta}_{BS2})\ d\hat{\theta}_{BS^1}\ d\hat{\theta}_{BS2}$

$+ \int_0^\infty \int_{\bar{R}} [\ \frac{m_1\hat{\theta}_{BS1_1}+m_2\hat{\theta}_{BS2}}{m_1+m_2} -$

$\theta]^2\ f_1\left(\hat{\theta}_{BS1}\right) f_2\left(\hat{\theta}_{BS2}\right) d\hat{\theta}_{BS1}\ d\hat{\theta}_{BS2}$

As utilization the same transformation :

$\mathrm{M} = \frac{(m_1-c+1)\theta}{\hat{\theta}_{\mathrm{BS1}}} \Rightarrow \hat{\theta}_{\mathrm{BS1}} = \frac{(m_1-c+1)\theta}{\mathrm{M}} \Rightarrow$

$d\ \hat{\theta}_{\mathrm{BS1}} = \frac{(m_1-c+1)\theta}{\mathrm{M}^2}$

$N = \frac{(m_2-c+1)\theta}{\hat{\theta}_{\mathrm{BS2}}} \Rightarrow \hat{\theta}_{\mathrm{BS2}} = \frac{(m_2-c+1)\theta}{\mathrm{M}} \Rightarrow$

$d\ \hat{\theta}_{\mathrm{BS2}} = \frac{(m_2-c+1)\theta}{\mathrm{N}^2}$

$MSE\ (\tilde{\theta}_{BSD2}) =$

$\frac{4(m_1-\mathrm{c}+1)^2\lambda^2+8(m_1-\mathrm{c}+1)\lambda^2\mathrm{r}_1+\lambda^2\mathrm{r}_1^2}{\mathrm{r}_2^2\,(m_1-1)!} \int_{R^*} m^{m_1-1}\ e^{-\mathrm{m}} dm\ -$

$\frac{8(m_1-\mathrm{c}+1)\lambda^3\mathrm{r}_1+4\lambda^3\mathrm{r}_1}{\mathrm{r}_2^2\,(m_1-1)!} \int_{R^*} m^{m_1-2}\ e^{-m} dm\ +$

$\frac{\mathrm{r}_1^2(m_1-\mathrm{c}+1)^2}{\mathrm{r}_2^2\,(m_1-1)!} \int_{R^*} m^{m_1-3}\ e^{-m} dm\ -$

$\frac{4(m_1-\mathrm{c}+1)^2\lambda\mathrm{r}_1+2(m_1-\mathrm{c}+1)\lambda\mathrm{r}_1^2}{\mathrm{r}_2^2\,(m_1-1)!} \int_{R^*} m^{m_1-2}\ e^{-m} dm\ -$

$\frac{2(1-\lambda)}{r_2}\ [\ (2(m_1 - \mathrm{c} + 1)\lambda +$

$r_1\lambda) \int_{R^*} \frac{m^{m_1-1}}{(m_1-1)!}\ e^{-m} dm\ -$

$2\lambda^2 \int_{R^*} \frac{m^{m_1-1}}{(m_1-1)!}\ e^{-m} dm\ - r_1(m_1 - \mathrm{c} +$

$1) \int_{R^*} m^{m_1-2}\ e^{-m} dm\ ](1-\lambda)^2\ \{I\ (\acute{r}_2, m_1) -$

$I\ (\acute{r}_1, m_1)\} + \frac{m_1^2(m_2-c+1)^2}{(m_1+m_2)^2(m_1-1)(m_1-2)}\ -$

$\frac{2m_1^2(m_2-c+1)}{(m_1+m_2)^2(m_1-1)}$

$+\frac{m_1^2}{(m_1+m_2)^2}+\frac{2m_1m_2-(2-c)^2}{(m_1+m_2)^2(m_1-1)(m_2-2)}+$

$\frac{m_2^2-(m_2-c+1)^2}{(m_1+m_2)^2(m_2-1)(m_2-2)} - \frac{2m_2^2-(m_2-c+1)}{(m_1+m_2)^2(m_2-1)} - \frac{m_2^2}{(m_1+m_2)^2} -$

$\frac{m_1^2-(m_1-c+1)^2}{(m_1+m_2)^2(m_1-1)(m_1-2)}\{I\ (\acute{r}_2, m_1-1) -$

$I\ (\acute{r}_1, m_1-1)\} - \frac{m_1^2}{(m_1+m_2)^2}\{I\ (\acute{r}_2, m_1) -$

$I\ (\acute{r}_1, m_1)\} - \frac{2m_1m_2-(2-c)}{(m_1+m_2)^2(m_2-1)}[\frac{m_1-c+1}{(m_1-1)}\{I\ (\acute{r}_2, m_1 -$

$1) - I\ (\acute{r}_1, m_1-1)\} - \{I\ (\acute{r}_2, m_1) - I\ (\acute{r}_1, m_1)\} -$

$\frac{m_1^2(m_2-c+1)^2}{(m_1+m_2)^2(m_2-1)(m_1-2)}\{I\ (\acute{r}_2, m_1) -$

$I\ (\acute{r}_1, m_1)\} + \frac{2m_2^2(m_2-c+1)}{(m_1+m_2)^2(m_2-1)}\{I\ (\acute{r}_2, m_1) -$

$I\ (\acute{r}_1, m_1)\} - \frac{m_1^2}{(m_1+m_2)^2}\ \{I\ (\acute{r}_2, m_1) - I\ (\acute{r}_1, m_1)\}$

(18)

3. Relative efficiency and Bais Ratio

To study the properties of the estimator $\tilde{\theta}_{BSD_1}$ & $\tilde{\theta}_{BSD_2}$, with respect $\tilde{\theta}_{Bs}$ estimator we must evaluate *Mean square error* for pooled Bayes estimator .

Therefore, obtain *Mean square error* on estimator $\tilde{\theta}_{Bs}$ as:

$$MSE(\tilde{\theta}_D) = E(\tilde{\theta}_D - \theta)^2 = \int_0^\infty \int_0^\infty (\hat{\theta}_D - \theta)^2 \ f_1(\hat{\theta}_{BS1}) f_2(\hat{\theta}_{BS2}) \, d\hat{\theta}_{BS1} \, d\hat{\theta}_{BS2}$$

By integration estimation , we obtain

$$MSE(\tilde{\theta}_D) = \theta^2 [\frac{m_1^2(m_1-c+1)^2}{(m_1+m_2)^2(m_1-1)(m_1-2)} - \frac{2m_1^2(m_1-c+1)}{(m_1+m_2)^2(m_1-1)} + \frac{m_1^2}{(m_1+m_2)^2} + \frac{2m_1m_2-(2-c)^2}{(m_1+m_2)^2(m_1-1)(m_2-2)} + \frac{m_2^2-(m_2-c+1)^2}{(m_1+m_2)^2(m_2-1)(m_2-2)} - \frac{2m_2^2-(m_2-c+1)}{(m_1+m_2)^2(m_2-1)} + \frac{m_2^2}{(m_1+m_2)^2}] \quad (19)$$

Thus ,the relative efficiency of the estimators $\tilde{\theta}_{BSDi}\ i = 1,2$ with cosidration to $\tilde{\theta}_D$ *is given by*

$$REE(\tilde{\theta}_{BSDi}) = \frac{MSE(\tilde{\theta}_D)}{MSE(\tilde{\theta}_{BSDi})} \ ; i = 1,2 \quad (20)$$

The findings indicate that the equation for the relative efficiency of the estimators $\tilde{\theta}_{BSDi}\ i = 1,2$ with respect to the Bayesian pooled estimator $\tilde{\theta}_{BD}$

We also define the bias ratio for the estimators $\tilde{\theta}_{BSDi}$ as :

$$BR(\tilde{\theta}_{BSDi}) = \frac{Bais(\tilde{\theta}_{BSDi} \mid \theta)}{\theta} \ ; i = 1,2 \quad (21)$$

NUMERICAL RESULTS

As Bias ratio and the relative efficiency of functions as (n , λ , α and c) to use with it values assumption

n_1= 5 , 10 , 15 n_2= 5 , 10 , 15
c = 1 , 2 , 3 K = 0.1 , 0.2

α = 0.01 , 0.05 λ = 0.3 , 0.6 , 0.9 , 1 , 1.3 , 1.6 , 1.9

As using the statical program we will get the following:

1. The proposed double Stage Shrinkage estimator $\tilde{\theta}_{BS1}$ and $\tilde{\theta}_{BS2}$ give high relative efficiency with respect classical pooled Bayes estimator in neighborhood λ=1, i.e. when $\theta = \theta_0$
2. The relative efficiency of the proposed double Stage Shrinkage of function for both estimator $\tilde{\theta}_{BS1}$ and $\tilde{\theta}_{BS2}$ with respect to classical pooled Bayes estimator $\tilde{\theta}_{BS}$ give highest value increasing function with "c=3"see Figure (1), and decreeing function of " c " see Figures (2)
3. From figures (3), (4) and (9) one can conclude that the relative efficiency of the proposed estimator $\tilde{\theta}_{BS1}$ and $\tilde{\theta}_{BS2}$ with respect to classical pooled Bayes estimator $\tilde{\theta}_{BS}$ is decreasing function of α and K.
4. It's clear from the figures (5), (6) the relative efficiency of the proposed estimator $\tilde{\theta}_{BS1}$ and $\tilde{\theta}_{BS2}$ with respect

to classical pooled Bayes estimator $\tilde{\theta}_{BS}$ is increasing function of n_1 .

5. The relative efficiency of the proposed estimator $\tilde{\theta}_{BS1}$ and $\tilde{\theta}_{BS2}$ with respect to classical pooled Bayes estimator $\tilde{\theta}_{BS}$ is decreasing function of n_2 see Figures (7) , (8)
6. The bias ratio of the both estimators $\tilde{\theta}_{BS1}$ and $\tilde{\theta}_{BS2}$ is reasonable in the neighborhood of λ between (0.3 to 1) off c and it is increasing when λ > 1 the estimators is decreeing as shown in Figures (10) while on Figures (11 , 12 , 13 , 17) as K , n_1 and α the both estimators $\tilde{\theta}_{BS1}$ and $\tilde{\theta}_{BS2}$ of λ between (0.3 to 1.9) is decreeing while appear on Figures (14 , 15) off n_2 is increasing but as shown in Figures (16) the bias ratio of the both estimators $\tilde{\theta}_{BS1}$ and $\tilde{\theta}_{BS2}$ is reasonable in the neighborhood of λ between (0.3 to 1.2) off c and it is increasing when λ > 1.2 the estimators is decreeing
7. The relative efficiency of the estimator $\tilde{\theta}_{BS1}$ with respect to classical pooled Bayes estimator $\tilde{\theta}_{BS}$ better than the relative efficiency of the estimator $\tilde{\theta}_{BS2}$ for λ between (0.3 to 1.3) while appear on Figure (18) when λ > 1.3 the estimator $\tilde{\theta}_{BS2}$ with respect to classical pooled Bayes estimator $\tilde{\theta}_{BS}$ better than the relative efficiency estimator $\tilde{\theta}_{BS1}$.

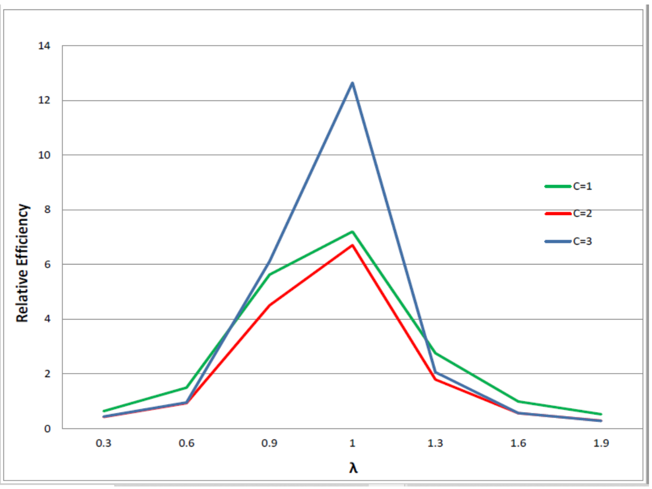

Figure (1) *show Relative efficiency of estimator θ ̃_(BS1), K=0.1, 0.2, c = 1 , 2 , 3*

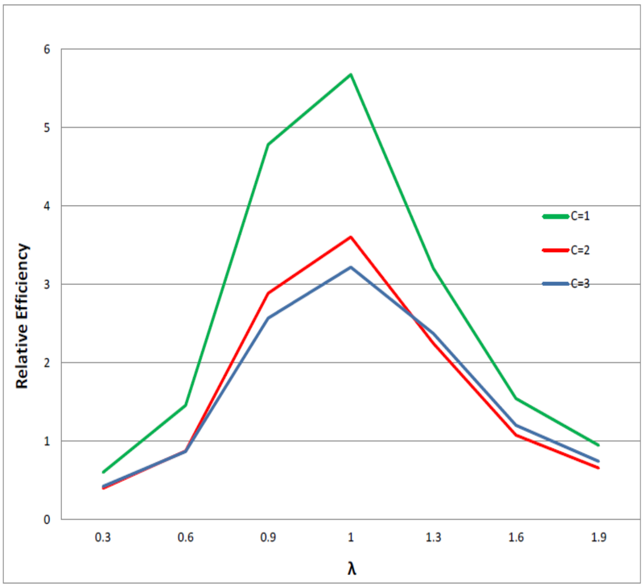

Figure (2) *show Relative efficiency of estimator θ ̃_(BS2), K=0.1, 0.2 , c= 1 , 2 , 3*

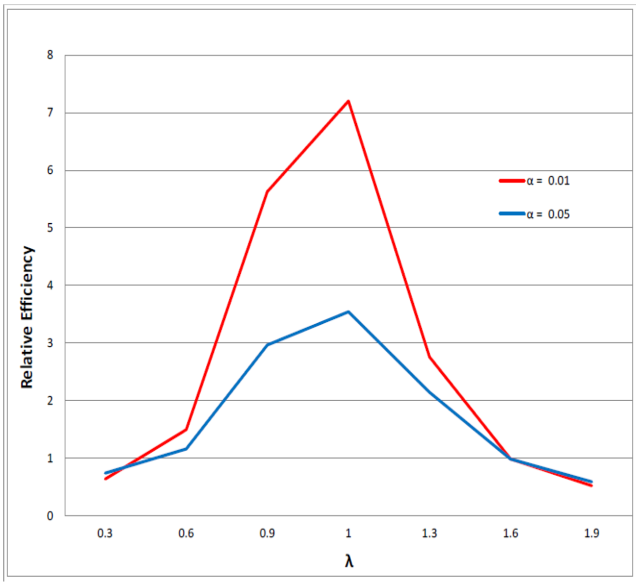

Figure (3) *show Relative efficiency of estimator* $\tilde{\theta}$*_(BS1), K=0.1, 0.2 , α= 0.01 , 0.05*

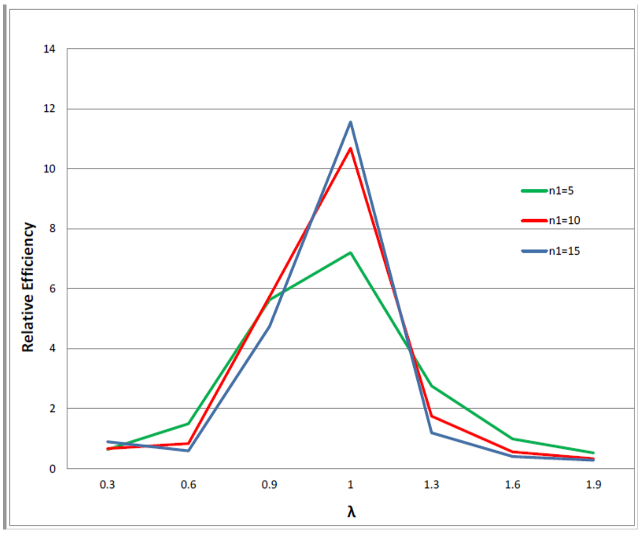

Figure (5) *show Relative efficiency of estimator* $\tilde{\theta}$*_(BS1), K=0.1, 0.2, n1 = 5 , 10 , 15*

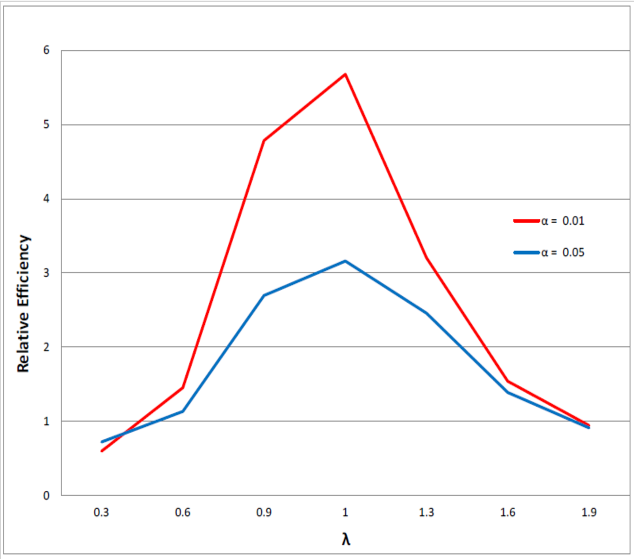

Figure (4) *show Relative efficiency of estimator* $\tilde{\theta}$*_(BS2), K=0.1, 0.2 , α= 0.01 , 0.05*

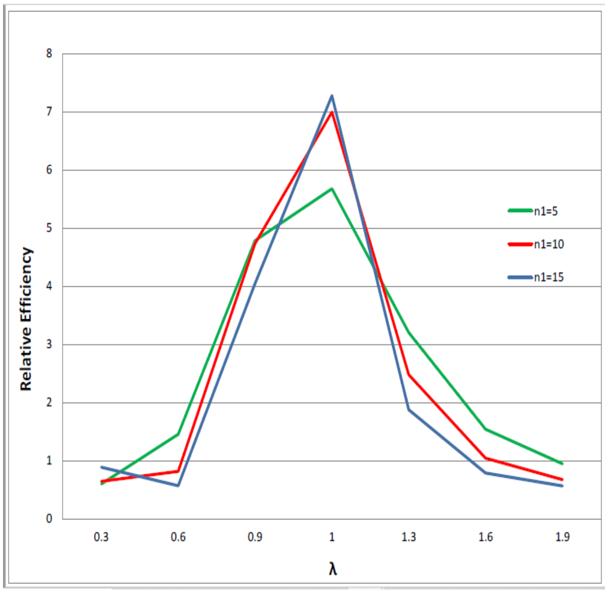

Figure (6) *show Relative efficiency of estimator* $\tilde{\theta}$*_(BS2), K=0.1, 0.2 , n1 = 5 , 10 , 15*

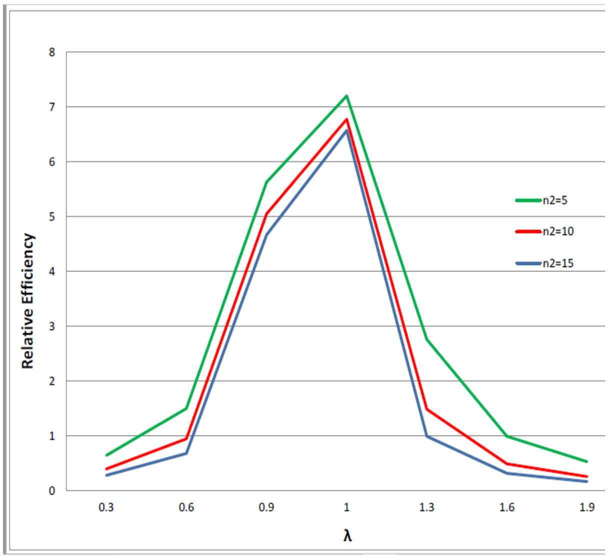

Figure (7) *show Relative efficiency of estimator $\tilde{\theta}$_(BS1), K=0.1, 0.2 , n2 = 5 , 10 ,15*

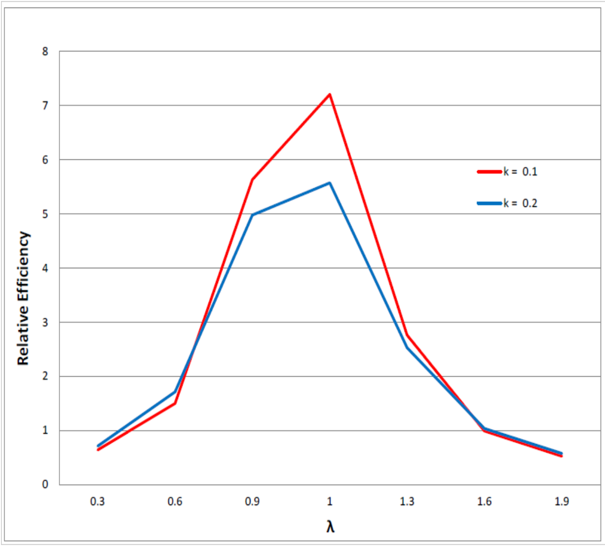

Figure (9) *show Relative efficiency of estimator $\tilde{\theta}$_(BS1), K=0.1, 0.2*

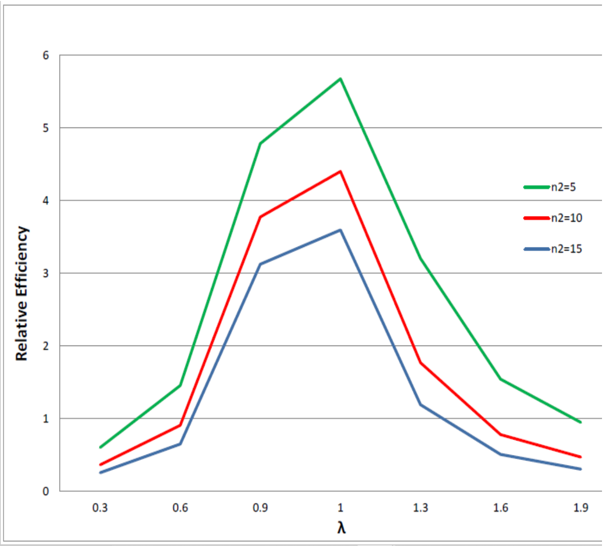

Figure (8) *show Relative efficiency of estimator $\tilde{\theta}$_(BS2), K=0.1, 0.2 , n2 = 5 , 10 ,15*

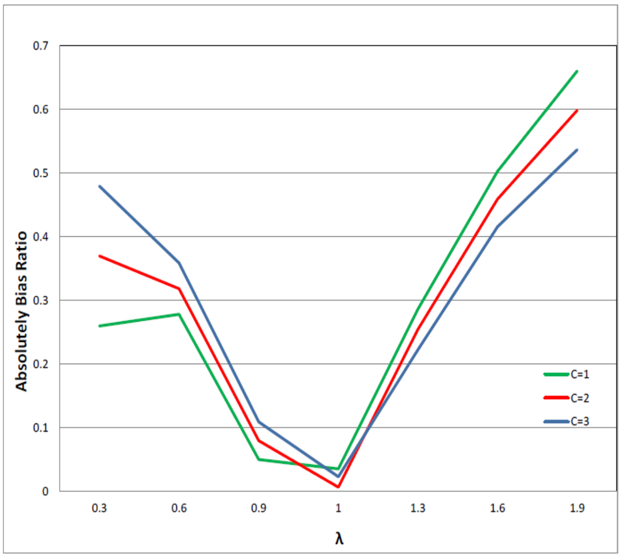

Figure (10) *show Bias Ratio of estimator $\tilde{\theta}$_(BS1), K=0.1 , c= 1 , 2, 3*

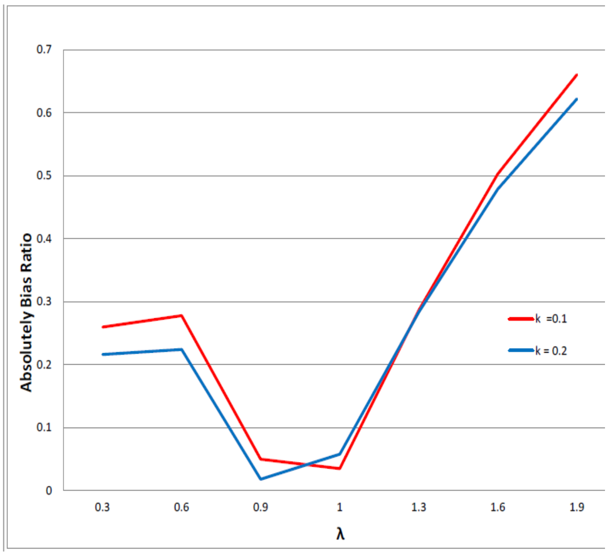

Figure (11) *show Bias Ratio of estimator $\tilde{\theta}$_(BS1), K=0.1, 0.2*

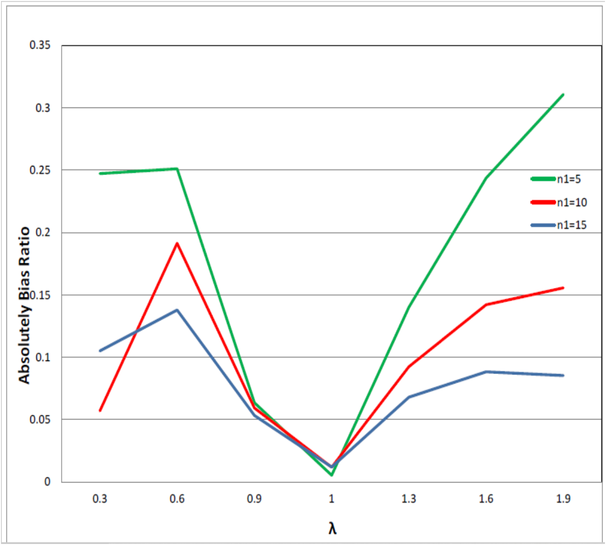

Figure (13) *show Bias Ratio of estimator $\tilde{\theta}$_(BS2), K=0.1, 0.2, n1 = 5,10,15*

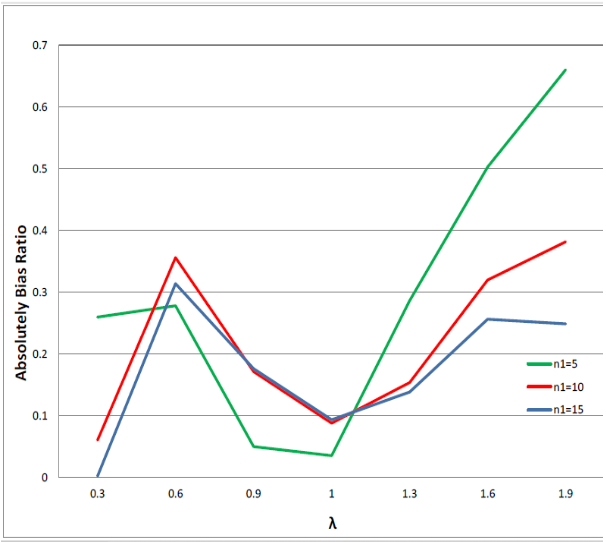

Figure (12) *show Bias Ratio of estimator $\tilde{\theta}$_(BS1), K=0.1, 0.2 , n1 = 5,10,15*

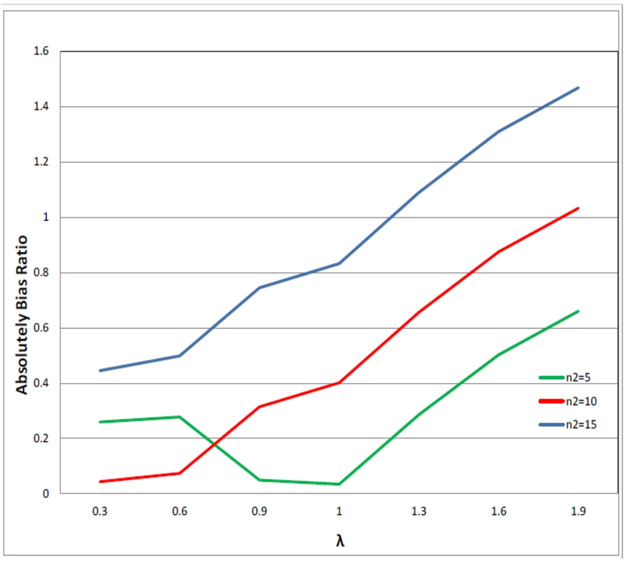

Figure (14) *show Bias Ratio of estimator $\tilde{\theta}$_(BS1), K=0.1, 0.2, n2 = 5,10,15*

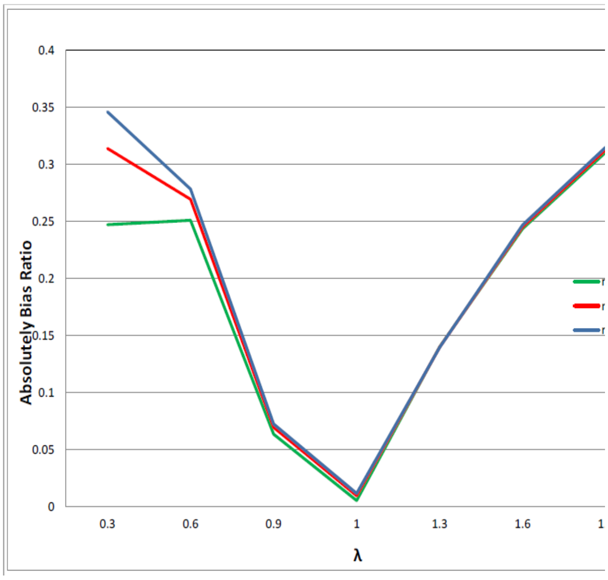

Figure (15) *show Bias Ratio of estimator* $\tilde{\theta}_{(BS2)}$*, K=0.1, 0.2, n2 = 5,10,15*

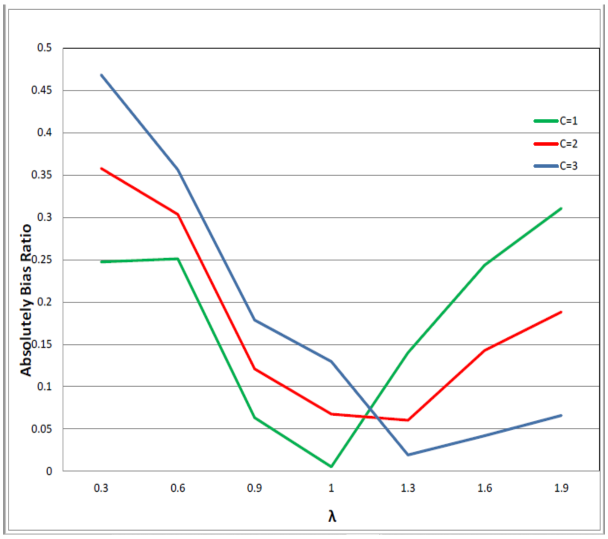

Figure (16) *show Bias Ratio of estimator* $\tilde{\theta}_{(BS2)}$*, K=0.1, 0.2, c = 1 , 2 , 3*

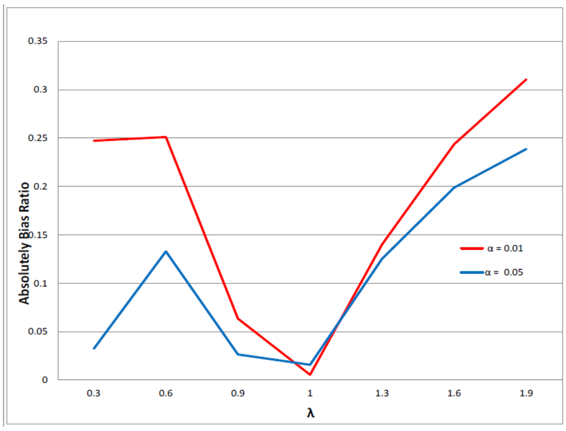

Figure (17) *show Bias Ratio of estimator* $\tilde{\theta}_{(BS2)}$*, K=0.1, 0.2, α = 0.01, 0.05*

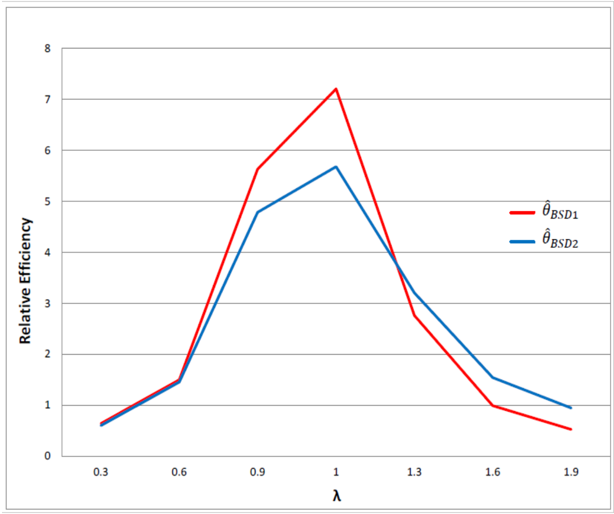

Figure (18) *show comparison between estimates* $\tilde{\theta}_{(BS1)}$*,* $\tilde{\theta}_{(BS2)}$

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