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An Investigation Into Some of The P-Valent Subclass That is Defined by That of The Generalized Derivative Operator

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Abstract

In this paper, introduce and study a subclass of multivalent functions that use the generalized derivative operator, There are a lot of things that can happen with coefficient inequalities growth and distortion, extreme points, radii that are very close to convexity, and more. star likeness and convexity for these subclasses.

Keywords: multivalent functions; Convolution (Hadamard products). There are a lot of things that can happen with coefficient inequalities, growth and distortion, extreme points, radii that are very close to convexity, and more.

INTRODUCTION

This is how it is shown. E(p) the class of the functions defied work with the form shown here:

$$f(z) = z^p + \sum_{h=p+1}^{\infty} a_h z^h \ , (z \in U, a_h \geq 0, p \in \mathbb{N} = \{1,2,3,\dots\}) \qquad (1.1)$$

which are analytic function and p-valent in the open unit disk U={z∈C:|z|<1}. Let B(p) denotes a subclass of E(p). Hadamard product (or convolution) (f*g) for two analytical f defined in equation (1.1) and g(z) given by

$$= z^p + \sum_{h=p+1}^{\infty} b_h z^h \ , \qquad (1.2)$$

which defined by

$$f(z) * g(z) = (f * g)(z) = z^p + \sum_{h=p+1}^{\infty} a_h b_h z^h \ . (z \in U, p \in \mathbb{N}) \ . \qquad (1.3)$$

A function f belong to the class E(p) is said to be multivalent starlike of order ∂, multivalent convex of order ∂ and multivalent closed-to-convex of order ∂,where (p∈N, ,0≤∂<p,z∈U), respectively if

$$Re\left\{\frac{z f'(z)}{f(z)}\right\} > \partial, Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \partial \ and \ Re\left\{\frac{f'(z)}{z^{p-1}}\right\} > \partial.$$

The following operator [was introduced by elhaddad and Darus 3]:

for the function f, belongs to the class E(p), we have:

$$\widetilde{\mathcal{D}}_{\lambda,p}^{m}(\upsilon,e,a_1,b_1)f(z) = z^p + \sum_{h=p+1}^{\infty}\left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Gamma(e)}{\Gamma(\upsilon(h-p)+e)}\left(\frac{\prod_{n=1}^{s}(a_n)_{h-p}}{\prod_{i=1}^{r}(b_i)_{h-p}}\right)\frac{a_h z^h}{(h-p)!}. \tag{1.4}$$

where a_n∈C,b_i∈C-{0,-1,-2,…},(i=1,….,r,n=1,…,s) and s≤r+1.

We write for the sake of simplicity

$$\widetilde{\mathcal{D}}_{\lambda,p}^{m}(\upsilon,e,a_1,b_1)f(z) = z^p + \sum_{j=p+1}^{\infty}\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,\upsilon,e)}(a_s,b_r)}{(j-p)!}a_j z^j, \tag{1.5}$$

〖where m∈N_0=N∪{0} ,λ≥0 and Y〗_((j-p,υ,e)) (a_s,b_r), is given by

$$\Upsilon_{(j-p,\upsilon,e)}(a_s,b_r) = \frac{\Gamma(e)}{\Gamma(\upsilon(j-p)+e)}\left(\frac{\prod_{n=1}^{s}(a_n)_{j-p}}{\prod_{i=1}^{r}(b_i)_{j-p}}\right). \tag{1.6}$$

See [2] for more detail on this operator.

Equation (1.5), we introduce class of the following class of analytical and multivalent functions.

Definition 1.1. A function f∈E(p) belongs in the class E(p,γ,λ) if

$$\left|\frac{(2-p)z[\widetilde{\mathcal{D}}_{\lambda,p}^{m}(\upsilon,e,a_1,b_1)f(z)]'' + z^2[\widetilde{\mathcal{D}}_{\lambda,p}^{m}(\upsilon,e,a_1,b_1)f(z)]'''}{(3\lambda-\gamma)z[\widetilde{\mathcal{D}}_{\lambda,p}^{m}(\upsilon,e,a_1,b_1)f(z)]'' + \lambda z^2[\widetilde{\mathcal{D}}_{\lambda,p}^{m}(\upsilon,e,a_1,b_1)f(z)]'''}\right| < 1, \tag{1.7}$$

where $\left(p \geq 1, \frac{1}{2} \leq \gamma < 1, 0 \leq \lambda \leq \frac{1}{2}\right), a_1 \in \mathbb{C}, b_1 \in \mathbb{C}\backslash\{0,-1,-2,\ldots\}, |z| < 1,$ $\upsilon, e \in \mathbb{C}, Re(\upsilon) > 0, Re(e) > 0, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$

Analytical and multivalent functions have been looked at by many different people over the years, help ensure that the coefficients are in for example, see Refs ([1], [2], [4], [5], [6], [7], [8], [9] and [10]). in this is an of work, we analyze and study go over for the class E(p,γ,λ), of analytic functions and multivalent. Also, coefficient bounds are some of the characteristics, the theorem of growth and distortion, inclusive properties, and extreme functions in our class are obtained.

GEOMETRIC PROPERTIES FOR E(p,γ,λ)

In this section, we present theorems with their evidence to address part of these geometric properties for such class E(p,γ,λ)

Theorem 2.1.A function f in equation (1.1) belongs to the class E(p,γ,λ) if and only i

$$\sum_{j=p+1}^{\infty}[p-j-\gamma+\lambda(j+1)]j(j-1)\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,\upsilon,e)}(a_s,b_r)}{(j-p)!}|a_j| \leq [\lambda(1+p)-\gamma(p-1)], \tag{2.1}$$

where $\left(p \geq 1, \frac{1}{2} \leq \gamma < 1, 0 < \lambda \leq \frac{1}{2}\right)$ *and* $\Upsilon_{(j-p,\upsilon,e)}(a_s,b_r)$ *is given by the equation (1.6).*

The result is sharp for the following function

$$f(z) = z^p + \frac{[\lambda(1+p)-\gamma]p(p-1)(j-p)!}{[p-j-\gamma+\lambda(j+1)]j(j-1)\Upsilon_{(j-p,\upsilon,e)}(a_s,b_r)\left[\frac{p+(j-p)\lambda}{p}\right]^m} z^j. \tag{2.2}$$

Proof: Suppose that $f \in E(p,\gamma,\lambda)$, then by the equation (1.7), we have:

$$\left|\frac{(2-p)z[\widetilde{\mathcal{D}}_{\lambda,p}^{m}(\upsilon,e,a_1,b_1)f(z)]'' + z^2[\widetilde{\mathcal{D}}_{\lambda,p}^{m}(\upsilon,e,a_1,b_1)f(z)]'''}{(3\lambda-\gamma)z[\widetilde{\mathcal{D}}_{\lambda,p}^{m}(\upsilon,e,a_1,b_1)f(z)]'' + \lambda z^2[\widetilde{\mathcal{D}}_{\lambda,p}^{m}(\upsilon,e,a_1,b_1)f(z)]'''}\right| < 1$$

$$= \left|(3\lambda-\gamma)z\left[\widetilde{\mathcal{D}}_{\lambda,p}^{m}(\upsilon,e,a_1,b_1)f(z)\right]'' + \lambda z^2\left[\widetilde{\mathcal{D}}_{\lambda,p}^{m}(\upsilon,e,a_1,b_1)f(z)\right]'''\right|$$

$$-\left|(2-p)z\left[\widetilde{\mathcal{D}}_{\lambda,p}^{m}(\upsilon,e,a_1,b_1)f(z)\right]'' + z^2\left[\widetilde{\mathcal{D}}_{\lambda,p}^{m}(\upsilon,e,a_1,b_1)f(z)\right]'''\right|.$$

$$= \left|(3\lambda-\gamma-2+p)z\left[\widetilde{\mathcal{D}}_{\lambda,p}^{m}(\upsilon,e,a_1,b_1)f(z)\right]''\right| + \left|(\lambda-1)z^2\left[\widetilde{\mathcal{D}}_{\lambda,p}^{m}(\upsilon,e,a_1,b_1)f(z)\right]'''\right|$$

$$= \left|(3\lambda-\gamma-2+p)z\left[p(p-1)z^{p-2} + h(h-1)\sum_{h=p+1}^{\infty}\left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)}{(h-p)!}a_h z^{h-2}\right]\right|$$

$$+\left|(\lambda-1)z^2\left[p(p-1)(p-2)z^{p-3} + h(h-1)(h-2)\sum_{h=p+1}^{\infty}\left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)}{(h-p)!}a_h z^{h-3}\right]\right|$$

$$= \left|(3\lambda-\gamma-2+p)p(p-1)z^{p-1} + (3\lambda-\gamma-2+p)h(h-1)\sum_{h=p+1}^{\infty}\left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)}{(h-p)!}a_h z^{h-1}\right|$$

$$+\left|(\lambda-1)p(p-1)(p-2)z^{p-1} + (\lambda-1)h(h-1)(h-2)\sum_{h=p+1}^{\infty}\left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)}{(h-p)!}a_h z^{h-1}\right|$$

$$= \left|[\lambda(1+p)-\gamma]p(p-1)z^{p-1} + \sum_{h=p+1}^{\infty}[p-h-\gamma+\lambda(h+1)]h(h-1)\left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)}{(h-p)!}a_h z^{h-1}\right|$$

$$\leq [\lambda(1+p)-\gamma]p(p-1)|z^{p-1}| + \sum_{h=p+1}^{\infty}[p-h-\gamma+\lambda(h+1)]h(h-1)\left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)}{(h-p)!}|a_h||z^{h-1}|$$

$$\leq |z^{p-1}| + \sum_{h=p+1}^{\infty}\frac{[p-h-\gamma+\lambda(h+1)]h(h-1)}{[\lambda(1+p)-\gamma]p(p-1)}\left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)}{(h-p)!}|a_h||z^{h-1}| \leq 1$$

$\lambda \geq 0, h > p, \frac{p+(h-p)\lambda}{p} > 0$, $(h-p)! > 0$ and $\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)$ is given by (1.6)

$$\sum_{h=p+1}^{\infty}[p-h-\gamma+\lambda(h+1)]h(h-1)\left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)}{(h-p)!}|a_h| \leq [\lambda(1+p)-\gamma]p(p-1).$$

Conversely, a suppose that equation (1.8) holds $|z| = s,\ \ s < 1$,then

$$\left|(3\lambda-\gamma)z\left[\widetilde{\mathcal{D}}_{\lambda,p}^{m}(\upsilon,e,a_1,b_1)f(z)\right]'' + \lambda z^2\left[\widetilde{\mathcal{D}}_{\lambda,p}^{m}(\upsilon,e,a_1,b_1)f(z)\right]'''\right|$$

$$-\left|(2-p)z\left[\widetilde{\mathcal{D}}_{\lambda,p}^{m}(\upsilon,e,a_1,b_1)f(z)\right]'' + z^2\left[\widetilde{\mathcal{D}}_{\lambda,p}^{m}(\upsilon,e,a_1,b_1)f(z)\right]'''\right|$$

$$= \left|(3\lambda-\gamma-2+p)z\left[\widetilde{\mathcal{D}}_{\lambda,p}^{m}(\upsilon,e,a_1,b_1)t(z)\right]''\right| + \left|(\lambda-1)z^2\left[\widetilde{\mathcal{D}}_{\lambda,p}^{m}(\upsilon,e,a_1,b_1)t(z)\right]'''\right|$$

$$= \left|(3\lambda-\gamma-2+p)z\left[p(p-1)z^{p-2} + h(h-1)\sum_{h=p+1}^{\infty}\left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)}{(h-p)!}a_h z^{h-2}\right]\right|$$

$$+\left|(\lambda-1)z^2\left[p(p-1)(p-2)z^{p-3} + h(h-1)(h-2)\sum_{h=p+1}^{\infty}\left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)}{(h-p)!}a_h z^{h-3}\right]\right|$$

$$= \left|(3\lambda-\gamma-2+p)p(p-1)z^{p-1}+(3\lambda-\gamma-2+p)h(h-1)\sum_{h=p+1}^{\infty}\left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,v,e)}(a_s,b_r)}{(h-p)!}a_h z^{h-1}\right|$$

$$+\left|(\lambda-1)p(p-1)(p-2)z^{p-1}+(\lambda-1)h(h-1)(h-2)\sum_{h=p+1}^{\infty}\left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,v,e)}(a_s,b_r)}{(h-p)!}a_h z^{h-1}\right|$$

$$= \left|[\lambda(1+p)-\gamma]p(p-1)z^{h-1}+\sum_{h=p+1}^{\infty}[p-h-\gamma+\lambda(h+1)]h(h-1)\left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,v,e)}(a_s,b_r)}{(h-p)!}a_h z^{h-1}\right|$$

$$\leq [\lambda(1+p)-\gamma]p(p-1)|z^{h-1}|+\sum_{h=p+1}^{\infty}[p-h-\gamma+\lambda(h+1)]h(h-1)\left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,v,e)}(a_s,b_r)}{(h-p)!}|a_h||z^{h-1}|$$

$$\leq |z^{p-1}|+\sum_{h=p+1}^{\infty}\frac{[p-h-\gamma+\lambda(h+1)]h(h-1)}{[\lambda(1+p)-\gamma]p(p-1)}\left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,v,e)}(a_s,b_r)}{(h-p)!}|a_h||z^{h-1}|$$

$$\sum_{h=p+1}^{\infty}[p-h-\gamma+\lambda(h+1)]h(h-1)\left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,v,e)}(a_s,b_r)}{(h-p)!}|a_h| \leq [\lambda(1+p)-\gamma]p(p-1)$$

where $|a_j|$ *is given by (1.8). So, we have:*

$$\sum_{h=p+1}^{\infty}[p-h-\gamma+\lambda(h+1)]h(h-1)\left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,v,e)}(a_s,b_r)}{(h-p)!}|a_h| - [\lambda(1+p)-\gamma]p(p-1) \leq 0$$

then $f \in E(p,\gamma,\lambda)$. the theorem is established .

Corollary 2.2. Let $f \in E(p,\gamma,\lambda)$. Then,

$$a_h \leq \frac{[\lambda(1+p)-\gamma]p(p-1)(h-p)!}{[p-h-\gamma+\lambda(h+1)]h(h-1)\Upsilon_{(h-p,v,e)}(a_s,b_r)\left[\frac{p+(h-p)\lambda}{p}\right]^m}, \tag{2.3}$$

where, ($h=p+1,p+2,\ldots$) $\left(p\geq 1,\frac{1}{2}\leq\gamma<1, 0<\lambda\leq\frac{1}{2}\right)$.

GROWTH AND DISTORTION THEOREMS

Bounds of |f(z)| and |f^' (z)| will be addressed by the following theorems respectively, where the bounds for multivalent function f(z) in the form

$$f(z) = z^p + \frac{[\lambda(1+p)-\gamma]p(p-1)}{[\lambda(p+2)-(1+\gamma)]p(1+p)\Upsilon_{(1,v,e)}(a_s,b_r)\left[\frac{p+\lambda}{p}\right]^m}z^{p+1}$$

$$s^p - s^{p+1}\frac{[\lambda(1+p)-\gamma]p(p-1)}{[\lambda(p+2)-(1+\gamma)]p(1+p)\Upsilon_{(1,v,e)}(a_s,b_r)\left[\frac{p+\lambda}{p}\right]^m} \leq |f(z)| \leq s^p + s^{p+1}\frac{[\lambda(1+p)-\gamma]p(p-1)}{[\lambda(p+2)-(1+\gamma)]p(1+p)\Upsilon_{(1,v,e)}(a_s,b_r)\left[\frac{p+\lambda}{p}\right]^m}$$

for $|z| = s, s < 1$.

Proof. As far as theorem (2.1) is concerned, we have

$$\sum_{h=p+1}^{\infty} \frac{[p-h-\gamma+\lambda(h+1)]h(h-1)}{[\lambda(1+p)-\gamma]p(p-1)} \left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,v,e)}(a_s,b_r)}{(h-p)!}|a_h| \le 1,$$

certain properties on analytical p-valent functions and

$$\frac{[\gamma-\lambda(h+1)]h(h-1)}{[\lambda(1+p)-\gamma]p(p-1)} \left[\frac{p+\lambda}{p}\right]^m \Upsilon_{(1,v,e)}(a_s,b_r) \sum_{h=p+1}^{\infty} |a_h|$$

$$\le \sum_{h=p+1}^{\infty} \frac{[p-h-\gamma+\lambda(h+1)]h(h-1)}{[\lambda(1+p)-\gamma]p(p-1)} \left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,v,e)}(a_s,b_r)}{(h-p)!}|a_h| \le 1,$$

so, we have

$$\sum_{h=p+1}^{\infty} |a_h| \le \frac{[\lambda(1+p)-\gamma]p(p-1)}{[\lambda(p+2)-(1+\gamma)]p(1+p)\Upsilon_{(1,v,e)}(a_s,b_r)\left[\frac{p+\lambda}{p}\right]^m}$$

From equation (1.1), we have

$$|f(z)| = \left|z^p + \sum_{h=p+1}^{\infty} a_h z^h\right| \le |z^p| + |z^{p+1}| \sum_{h=p+1}^{\infty} |a_h| \le s^p + s^{p+1} \sum_{h=p+1}^{\infty} |a_h|$$
$$\le s^p + \frac{[\lambda(1+p)-\gamma]p(p-1)}{[\lambda(p+2)-(1+\gamma)]p(1+p)\Upsilon_{(1,v,e)}(a_s,b_r)\left[\frac{p+\lambda}{p}\right]^m} s^{p+1}$$

And this is how other arguments can be proven.

$$|f(z)| = \left|z^p + \sum_{h=p+1}^{\infty} a_h z^h\right| \ge |z^p| - |z^{p+1}| \sum_{h=p+1}^{\infty} |a_h| \ge s^p - s^{p+1} \sum_{h=p+1}^{\infty} |a_h|$$
$$\ge s^p - \frac{[\lambda(1+p)-\gamma]p(p-1)}{[\lambda(p+2)-(1+\gamma)]p(1+p)\Upsilon_{(1,v,e)}(a_s,b_r)\left[\frac{p+\lambda}{p}\right]^m} s^{p+1}$$

proof is complete.

Theorem 3.2. *If* $f \in E(p,\gamma,\lambda)$, *then for* $|z| = s, s < 1$, *we have*

$$ps^{p-1} - \frac{[\lambda(1+p)-\gamma]p(p-1)}{[\lambda(p+2)-(1+\gamma)]p(1+p)\Upsilon_{(1,v,e)}(a_s,b_r)\left[\frac{p+\lambda}{p}\right]^m} s^p \le |f'(z)|$$
$$\le ps^{p-1} + \frac{[\lambda(1+p)-\gamma]p(p-1)}{[\lambda(p+2)-(1+\gamma)]p(1+p)\Upsilon_{(1,v,e)}(a_s,b_r)\left[\frac{p+\lambda}{p}\right]^m} s^p$$

proof: Let $f \in F_{v,e}^m(a_s,b_r,\lambda;h,p)$, then from equation (2.1), we have

$$\sum_{h=p+1}^{\infty} |a_h| \le \frac{[\lambda(1+p)-\gamma]p(p-1)}{[\lambda(p+2)-(1+\gamma)]p(1+p)\Upsilon_{(1,v,e)}(a_s,b_r)\left[\frac{p+\lambda}{p}\right]^m}.$$

Also, from equation (1.1), we have

$$|f'(z)| = \left|pz^{p-1} + \sum_{h=p+1}^{\infty} h a_h z^{h-1}\right| \le ps^{p-1} + (p+1)s^p \sum_{h=p+1}^{\infty} |a_h|$$

$$\leq ps^{p-1} + \frac{[\lambda(1+p)-\gamma]p(p-1)}{[\lambda(p+2)-(1+\gamma)]p(1+p)\Upsilon_{(1,\upsilon,e)}(a_s,b_r)\left[\frac{p+\lambda}{p}\right]^m} s^p$$

And this is how other arguments can be proven.

$$|f'(z)| = \left|pz^{p-1} + \sum_{h=p+1}^{\infty} ha_h z^{h-1}\right| \geq ps^{p-1} - (p+1)s^p \sum_{h=p+1}^{\infty} |a_h|$$

$$\geq ps^{p-1} - \frac{[\lambda(1+p)-\gamma]p(p-1)}{[\lambda(p+2)-(1+\gamma)]p(1+p)\Upsilon_{(1,\upsilon,e)}(a_s,b_r)\left[\frac{p+\lambda}{p}\right]^m} s^p$$

The proof is complete.

RADII OF STARLIKENESS, CONVEXITY AND CLOSE-TO-CONVEXITY

The radii of starlikeness, convexity, and close to convexity will also be used to change the following theorems into new ones.

Theorem 4.1.If the function does what it says it does, f(z) belongs a follower of class E(p,γ,λ) as shown in the figure the equation (1.2).

Then it is multivalent starlike of order $\partial(0\leq\partial<p)$in the open disk |z|≤s_1, such that

$s_1(p,\gamma,\lambda,\partial)$

$$= inf_h\left[\sum_{h=p+1}^{\infty} \frac{(p-\partial)[p-h-\gamma+\lambda(h+1)]h(h-1)}{(h-\partial)[\lambda(1+p)-\gamma]p(p-1)}\left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)}{(h-p)!}\right]^{\frac{1}{h-p}}$$

$$(h \geq p+1).$$

The result is very good for the outside function f(z) given by equation (2.2).

Proof: It's enough to show this is true

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq p - \partial, \text{ where } (0 \leq \partial < p),$$

for $|z| < s_1(p,\gamma,\lambda,\partial)$, we have

$$\left|\frac{zf'(z)}{f(z)} - p\right| = \left|\frac{z\left[pz^{p-1} + \sum_{h=p+1}^{\infty} ha_h z^{h-1}\right] - p\left[z^p + \sum_{h=p+1}^{\infty} a_h z^h\right]}{z^p + \sum_{h=p+1}^{\infty} a_h z^h}\right|$$

$$= \left|\frac{\left[\sum_{h=p+1}^{\infty} ha_h z^h\right] - p\left[\sum_{h=p+1}^{\infty} a_h z^h\right]}{z^p + \sum_{h=p+1}^{\infty} a_h z^h}\right| \leq \frac{\left[\sum_{h=p+1}^{\infty}(h-p)|a_h||z|^{h-p}\right]}{\left[1-\sum_{h=p+1}^{\infty}|a_h||z|^{h-p}\right]}$$

Thus $\left|\frac{zf'(z)}{f(z)} - p\right| \leq p - \partial.$

If $\sum_{h=p+1}^{\infty} \frac{(h-\partial)a_h|z|^{h-p}}{(p-\partial)} \leq 1.$

Therefore by Corollary (2.2), the last inequality is valid, if it is true,

$$\frac{(h-\partial)|z|^{h-p}}{(p-\partial)} \leq \frac{[p-h-\gamma+\lambda(h+1)]h(h-1)}{[\lambda(1+p)-\gamma]p(p-1)}\left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)}{(h-p)!}$$

equivalently if:

$$|z| \leq \left[\frac{(p-\partial)[p-h-\gamma+\lambda(h+1)]h(h-1)}{(h-\partial)[\lambda(1+p)-\gamma]p(p-1)}\left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)}{(h-p)!}\right]^{\frac{1}{h-p}} \quad (4.1)$$

Theorem immediately follows from (4.1)

Theorem 4.2.If the function does what it says it does f(z) belongs a follower of class E(p,γ,λ) as shown in the figure the equation (1.2).

Then f(z) is multivalent convex of ∂ (0≤∂<p) in the disk that is open |z|<s_2, were

$$s_2(p,\gamma,\lambda,\partial)$$
$$= inf_h \left[\frac{(p-\partial)[p-h-\gamma+\lambda(h+1)](h-1)}{(h-\partial)[\lambda(1+p)-\gamma]p(p-1)}\left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)}{(h-p)!}\right]^{\frac{1}{h-p}} \quad (h \geq p+1),$$

for the result is clear for the outside function f(z) given by equation (2.2).

Proof: It's enough to show that

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right| \leq p-\partial, \qquad where\ (0 \leq \partial < p),$$

for $|z| < s_2(p,\gamma,\lambda,\partial)$, we have

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right|$$
$$= \left|1+\frac{z\left[p(p-1)z^{p-2}+\sum_{h=p+1}^{\infty} h(h-1)a_h z^{h-2}\right]-p\left[pz^{p-1}+\sum_{h=p+1}^{\infty} ha_h z^{h-1}\right]}{\left[pz^{p-1}+\sum_{h=p+1}^{\infty} ha_h z^{h-1}\right]}\right|$$
$$= \left|\frac{\left[\sum_{h=p+1}^{\infty} h^2 a_h z^{h-1}\right]-\left[ph\sum_{h=p+1}^{\infty} a_h z^{h-1}\right]}{\left[pz^{p-1}+\sum_{h=p+1}^{\infty} ha_h z^{h-1}\right]}\right|$$
$$\left|1+\frac{zf''(z)}{f'(z)}-p\right| \leq \frac{\left[\sum_{h=p+1}^{\infty} h(h-p)a_h|z|^{h-p}\right]}{\left[1-\sum_{h=p+1}^{\infty} ha_h|z|^{h-p}\right]}\ .$$

Thus $\left|1+\frac{zf''(z)}{f'(z)}-p\right| \leq p-\partial$,

if $\displaystyle\sum_{h=p+1}^{\infty} \frac{h(h-\partial)a_h|z|^{h-p}}{(p-\partial)} \leq 1.$

Thus, by Corollary (2.2), the last inequality is valid if it is true :

$$\frac{h(h-\partial)|z|^{h-p}}{(p-\partial)} \leq \frac{[p-h-\gamma+\lambda(h+1)]h(h-1)}{[\lambda(1+p)-\gamma]p(p-1)}\left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)}{(h-p)!}$$

equivalently if

$$|z| \leq \left[\frac{(p-\partial)[p-h-\gamma+\lambda(h+1)](h-1)}{(h-\partial)[\lambda(1+p)-\gamma]p(p-1)}\left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)}{(h-p)!}\right]^{\frac{1}{h-p}} \tag{4.2}$$

theorem immediately follows from (4.2) .

Theorem 4.3.Let the function f(z) defined by equation (1.2) be the class E(p,γ,λ).

Then f(z) is multivalent an order that is close-to-convex ∂ (0≤∂<p) in open disk |z|<s_3, such that.

$$s_3(p,\gamma,\lambda,\partial)$$
$$= inf_h \left[\frac{(p-\partial)[p-h-\gamma+\lambda(h+1)](h-1)}{[\lambda(1+p)-\gamma]p(p-1)}\left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)}{(h-p)!}\right]^{\frac{1}{h-p}} . \ (h \geq p+1)$$

for the result is clear for external function f(z) given by the equation (2.2).

Proof: It's enough to show that

$$\left|\frac{f'(z)}{z^{p-1}}-p\right| \leq p-\partial\ , \qquad where\ (0 \leq \partial < p),$$

for $|z| < s_3(p,\gamma,\lambda,\partial)$, we have

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| = \left|\frac{pz^{p-1} + \sum_{h=p+1}^{\infty} ha_h z^{h-1} - pz^{p-1}}{z^{p-1}}\right| = \left|\sum_{h=p+1}^{\infty} ha_h z^{h-p}\right|$$

$$\leq \sum_{h=p+1}^{\infty} ha_h |z|^{h-p}.$$

Thus $\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p - \partial$.

If $\sum_{h=p+1}^{\infty} \frac{ha_h |z|^{h-p}}{(p-\partial)} \leq 1$.

Thus, by Corollary (2.2), the last inequality is valid if it is true

$$\frac{h|z|^{h-p}}{(p-\partial)} \leq \frac{[p-h-\gamma+\lambda(h+1)]h(h-1)}{[\lambda(1+p)-\gamma]p(p-1)} \left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)}{(h-p)!}$$

equivalently if

$$|z| \leq \left[\frac{(p-\partial)[p-h-\gamma+\lambda(h+1)](h-1)}{[\lambda(1+p)-\gamma]p(p-1)} \left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)}{(h-p)!}\right]^{\frac{1}{h-p}}. \tag{4.3}$$

Theorem comes right away from the fact in the equation (4.3) .

EXTREME POINTS

The theorem below addresses points at the end of the class E(p,γ,λ).

Theorem 5.1. Let f_p (z)=z^p an

$$f_h(z) = z^p + \sum_{h=p+1}^{\infty} \frac{[p-h-\gamma+\lambda(h+1)]h(h-1)}{[\lambda(1+p)-\gamma]p(p-1)} \left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)}{(h-p)!} z^h$$

where $\left(h \geq p+1, p \geq 1, \frac{1}{2} \leq \gamma < 1, 0 < \lambda \leq \frac{1}{2}\right)$.

Then a function f(z) belongs to a certain class $E(p,\gamma,\lambda)$ if, and only if, this is true be written as:

$$f(z) = \mathcal{L}_p z^p + \sum_{h=p+1}^{\infty} \mathcal{L}_h f_h(z), \tag{5.1}$$

such that

$$(\mathcal{L}_p \geq 0, \mathcal{L}_h \geq 0, h \geq p+1)\ and\ \mathcal{L}_p + \sum_{h=p+1}^{\infty} \mathcal{L}_h = 1$$

proof: Suppose that $f(z)$ thus defined in equation (5.1), then

$$\mathrm{f(z)} = \mathcal{L}_\mathrm{p} \mathrm{z}^\mathrm{p} + \sum_{\mathrm{h=p+1}}^{\infty} \mathcal{L}_\mathrm{h} \left[\mathrm{z}^\mathrm{p} + \frac{[\lambda(1+\mathrm{p})-\gamma]\mathrm{p}(\mathrm{p}-1)}{[\mathrm{p}-\mathrm{h}-\gamma+\lambda(\mathrm{h}+1)]\mathrm{h}(\mathrm{h}-1)} \left[\frac{\mathrm{p}}{\mathrm{p}+(\mathrm{h}-\mathrm{p})\lambda}\right]^\mathrm{m} \frac{(\mathrm{h}-\mathrm{p})!}{\Upsilon_{(\mathrm{h-p},\upsilon,\mathrm{e})}(\mathrm{a_s},\mathrm{b_r})} \mathrm{z}^\mathrm{h}\right]$$

$$= \mathrm{z}^\mathrm{p} + \sum_{\mathrm{h=p+1}}^{\infty} \left[\frac{[\lambda(1+\mathrm{p})-\gamma]\mathrm{p}(\mathrm{p}-1)}{[\mathrm{p}-\mathrm{h}-\gamma+\lambda(\mathrm{h}+1)]\mathrm{h}(\mathrm{h}-1)} \left[\frac{\mathrm{p}}{\mathrm{p}+(\mathrm{h}-\mathrm{p})\lambda}\right]^\mathrm{m} \frac{(\mathrm{h}-\mathrm{p})!}{\Upsilon_{(\mathrm{h-p},\upsilon,\mathrm{e})}(\mathrm{a_s},\mathrm{b_r})}\right] \mathcal{L}_\mathrm{h} \mathrm{z}^\mathrm{h}$$

hence

$$\sum_{h=p+1}^{\infty}\left[\frac{[\lambda(1+p)-\gamma]p(p-1)}{[p-h-\gamma+\lambda(h+1)]h(h-1)}\left[\frac{p}{p+(h-p)\lambda}\right]^{m}\frac{(h-p)!}{\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)}\right]$$
$$\times\left[\frac{[p-h-\gamma+\lambda(h+1)]h(h-1)}{[\lambda(1+p)-\gamma]p(p-1)}\left[\frac{p+(h-p)\lambda}{p}\right]^{m}\frac{\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)}{(h-p)!}\right]\mathcal{L}_h$$
$$=\sum_{h=p+1}^{\infty}\mathcal{L}_h=1-\mathcal{L}_p\leq 1\,.$$

Thus $f\in B(p,\gamma,\lambda)$.

Conversely, suppose that $f(z)\in E(p,\gamma,\lambda)$ we may be setting

$$\mathcal{L}_h=\sum_{h=p+1}^{\infty}\left[\frac{[p-h-\gamma+\lambda(h+1)]h(h-1)}{[\lambda(1+p)-\gamma]p(p-1)}\left[\frac{p+(h-p)\lambda}{p}\right]^{m}\frac{\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)}{(h-p)!}\right]a_h,$$

where a_h is defined in equation (2.1)

then $f(z)=z^p+\sum_{h=p+1}^{\infty}a_h z^h$

$$=z^p+\sum_{h=p+1}^{\infty}\left[\frac{[\lambda(1+p)-\gamma]p(p-1)}{[p-h-\gamma+\lambda(h+1)]h(h-1)}\left[\frac{p}{p+(h-p)\lambda}\right]^{m}\frac{(h-p)!}{\Upsilon_{(h-p,v,e)}(a_s,b_r)}\right]\mathcal{L}_h z^h$$
$$=z^p+\sum_{h=p+1}^{\infty}[f_h(z)-z^p]\;=\sum_{h=p+1}^{\infty}\mathcal{L}_h f_h(z)+(1-\sum_{h=p+1}^{\infty}\mathcal{L}_h)z^p$$

thus $f(z)=\mathcal{L}_p z^p+\sum_{h=p+1}^{\infty}\mathcal{L}_h f_h(z)$

The proof is all completed now (2.1).

CONVOLUTION PROPERTIES

The following theorems explain the properties of a functions in class E(p,γ,λ) that make them convolution.

Theorem 6.1.Let the functions f_s (z)belongs to the class E(p,γ,λ) such that

$$f_s(z)=z^p+\sum_{h=p+1}^{\infty}a_{h,s}z^h\,,\qquad \left(a_{h,s}\geq 0,s=1,2\right)\tag{6.1}$$

Then $(f_1*f_2)\in E(p,\gamma,k)$, where

$$k\geq\frac{p(p-1)[p-h-\gamma][(1+\lambda p)-\gamma]^2p^m(h-p)!+h(h-1)\gamma[p-h-\gamma+(\lambda h+1)]^2[p+(h-p)\lambda]^m\Upsilon_{(h-p,v,e)}}{(1+p)h(h-1)[p-h-\gamma+(\lambda h+1)]^2[p+(h-p)\lambda]^m\Upsilon_{(h-p,v,e)}(a_s,b_r)-(h+1)p(p-1)[(1+\lambda p)-\gamma]^2p^m}$$

This is a very good result for the functions that were f_s (s=1,2) given by equation (2.2), $k\in\mathbb{C}/\{0\}$.

Proof. we will find the smallest k such that

$$\sum_{h=p+1}^{\infty}\left[\frac{[p-h-\gamma+k(h+1)]h(h-1)}{[k(1+p)-\gamma]p(p-1)}\left[\frac{p+(h-p)\lambda}{p}\right]^{m}\frac{\Upsilon_{(h-p,v,e)}(a_s,b_r)}{(h-p)!}\right]a_{h,1}a_{h,2}\leq 1$$

since $f_s\in E(p,\gamma,\lambda),(s=1,2),$ then

$$\sum_{h=p+1}^{\infty}\left[\frac{[p-h-\gamma+(\lambda h+1)]h(h-1)}{[(1+\lambda p)-\gamma]p(p-1)}\left[\frac{p+(h-p)\lambda}{p}\right]^{m}\frac{\Upsilon_{(h-p,v,e)}(a_s,b_r)}{(h-p)!}\right]a_{h,s}\leq 1,\;\;(s=1,2)$$

By Cauhy –Schwarz inequality, we get

$$\sum_{j=p+1}^{\infty}\left[\frac{[p-j-\gamma+(\lambda j+1)]j(j-1)}{[(1+\lambda p)-\gamma]p(p-1)}\left[\frac{p+(j-p)\lambda}{p}\right]^{m}\frac{\Upsilon_{(j-p,\upsilon,e)}(a_s,b_r)}{(j-p)!}\right]\sqrt{a_{j,1}a_{j,2}} \leq 1, \qquad (6.2)$$

Now, the only thing we need to prove is that:

$$\left[\frac{[p-h-\gamma+k(h+1)]h(h-1)}{[k(1+p)-\gamma]p(p-1)}\left[\frac{p+(h-p)\lambda}{p}\right]^{m}\frac{\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)}{(h-p)!}\right]a_{h,1}a_{h,2}$$

$$\leq \left[\frac{[p-h-\gamma+(\lambda h+1)]h(h-1)}{[(1+\lambda p)-\gamma]p(p-1)}\left[\frac{p+(h-p)\lambda}{p}\right]^{m}\frac{\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)}{(h-p)!}\right]\sqrt{a_{h,1}a_{h,2}}$$

and equivalently to:

$$\sqrt{a_{h,1}a_{h,2}} \leq \frac{[k(1+p)-\gamma][p-h-\gamma+(\lambda h+1)]}{[p-h-\gamma+k(h+1)][(1+\lambda p)-\gamma]}$$

from equation (6.2) we have

$$\sqrt{a_{h,1}a_{h,2}} \leq \frac{1}{\frac{[p-h-\gamma+(\lambda h+1)]h(h-1)}{[(1+\lambda p)-\gamma]p(p-1)}\left[\frac{p+(h-p)\lambda}{p}\right]^{m}\frac{\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)}{(h-p)!}}$$

This ends well enough to illustrate that

$$\frac{[(1+\lambda p)-\gamma]p(p-1)\,p^{m}(h-p)!}{[p-h-\gamma+(\lambda h+1)]h(h-1)[p+(h-p)\lambda]^{m}\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)}$$

$$\leq \frac{[k(1+p)-\gamma][p-h-\gamma+(\lambda h+1)]}{[p-h-\gamma+k(h+1)][(1+\lambda p)-\gamma]}, \quad \text{then}$$

$$[p-h-\gamma][(1+\lambda p)-\gamma]^{2}p(p-1)p^{m}(h-p)!+k(h+1)[(1+\lambda p)-\gamma]^{2}p(p-1)p^{m}(h-p)!$$

$$\leq (-\gamma)[p-h-\gamma+(\lambda h+1)]^{2}h(h-1)[p+(h-p)\lambda]^{m}\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)$$

$$+k(1+p)[p-h-\gamma+(\lambda h+1)]^{2}h(h-1)[p+(h-p)\lambda]^{m}\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)$$

$$k$$

$$\geq \frac{p(p-1)[p-h-\gamma][(1+\lambda p)-\gamma]^{2}p^{m}(h-p)!+h(h-1)\gamma[p-h-\gamma+(\lambda h+1)]^{2}[p+(h-p)\lambda]^{m}\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)}{(1+p)h(h-1)[p-h-\gamma+(\lambda h+1)]^{2}[p+(h-p)\lambda]^{m}\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)-(h+1)p(p-1)[(1+\lambda p)-\gamma]^{2}p^{m}(h-p)!}$$

Thus, the theorem is established .

Theorem 6.2. Let the functions $f_s(z)$ in theorem (6.1) belongs of a class $E(p,\gamma,\lambda)$,then a function $f(z)=z^{p}+\sum_{h=p+1}^{\infty}(a_{h,1}^{2}+a_{h,2}^{2})z^{h}$ belongs also of a class $E(p,\gamma,\lambda)$ *where* $p(p+1)[1-(\lambda(p+1)+1)+\gamma]-2p(p-1)[(\lambda p+1)-\gamma]\geq 0.$

proof: since $f_1(z)\in E(p,\gamma,\lambda)$, we get

$$\sum_{h=p+1}^{\infty}\left[\frac{[p-h-\gamma+(\lambda h+1)]h(h-1)}{[(1+\lambda p)-\gamma]p(p-1)}\left[\frac{p+(h-p)\lambda}{p}\right]^{m}\frac{\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)}{(h-p)!}\right]^{2}a_{h,1}^{2}$$

where $(h\geq p+1, p\geq 1, \frac{1}{2}\leq\gamma<1, 0<\lambda\leq\frac{1}{2})$

$$\leq\left(\sum_{h=p+1}^{\infty}\left[\frac{[p-h-\gamma+(\lambda h+1)]h(h-1)}{[(1+\lambda p)-\gamma]p(p-1)}\left[\frac{p+(h-p)\lambda}{p}\right]^{m}\frac{\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)}{(h-p)!}\right]a_{h,1}\right)^{2}\leq 1 \qquad (6.3)$$

and

$$\sum_{h=p+1}^{\infty}\left[\frac{[p-h-\gamma+(\lambda h+1)]h(h-1)}{[(1+\lambda p)-\gamma]p(p-1)}\left[\frac{p+(h-p)\lambda}{p}\right]^{m}\frac{\Upsilon_{(h-p,\upsilon,e)}(a_s,b_r)}{(h-p)!}\right]^{2}a_{h,2}^{2}$$

$$\leq \left(\sum_{h=p+1}^{\infty} \left[\frac{[p-h-\gamma+(\lambda h+1)]h(h-1)}{[(1+\lambda p)-\gamma]p(p-1)} \left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,v,e)}(a_s,b_r)}{(h-p)!} \right] a_{h,2} \right)^2 \leq 1, \qquad (6.4)$$

the inequalities equation (6.3) and equation (6.4) gives

$$\sum_{h=p+1}^{\infty} \frac{1}{2} \left[\frac{[p-h-\gamma+(\lambda h+1)]h(h-1)}{[(1+\lambda p)-\gamma]p(p-1)} \left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,v,e)}(a_s,b_r)}{(h-p)!} \right]^2 \left(a_{h,1}^2 + a_{h,2}^2\right) \leq 1$$

According to theorem (2.1), it's enough to show that

$$\sum_{h=p+1}^{\infty} \left[\frac{[p-h-\gamma+(\lambda h+1)]h(h-1)}{[(1+\lambda p)-\gamma]p(p-1)} \left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,v,e)}(a_s,b_r)}{(h-p)!} \right] \left(a_{h,1}^2 + a_{h,2}^2\right) \leq 1.$$

Thus, if the last inequality is achieved, (h=p+1, p+2, p+3, …)

$$\left[\frac{[p-h-\gamma+(\lambda h+1)]h(h-1)}{[(1+\lambda p)-\gamma]p(p-1)} \left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,v,e)}(a_s,b_r)}{(h-p)!} \right]$$

$$\leq \frac{1}{2} \left[\frac{[p-h-\gamma+(\lambda h+1)]h(h-1)}{[(1+\lambda p)-\gamma]p(p-1)} \left[\frac{p+(h-p)\lambda}{p}\right]^m \frac{\Upsilon_{(h-p,v,e)}(a_s,b_r)}{(h-p)!} \right]^2.$$

Or, if

$$[(1+\lambda p)-\gamma]p(p-1) - 2h(h-1)[p-h-\gamma+(\lambda h+1)] \geq 0 \qquad (6.5)$$

for $h = p+1, p+2, p+3, \dots$.

A left-hand side of an equation (6.5) is increasing the function of h, so it is satisfied for all h,

$p(p+1)[\lambda(p+1)-\gamma] - 2[(1+\lambda p)-\gamma]p(p-1) \geq 0$

which is true by our assumption therefor the prove is complete .

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