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Describing Determinants

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Abstract
The evaluation of determinant of orders 2 × 2 or 3 × 3 is relatively straightforward. But if we consider determinant of higher orders say *n* × *n* where $n \geq 4$ it is not that easy. In this paper, we shall see a method from which we can determine the value of a particular *n* × *n* determinant using just three numbers whose answer turns to be very interesting numbers in mathematics. The asymptotic relations concerning the *n* × *n* determinant is also obtained in this paper.

Keywords: Determinant, Recurrence Relation, Mathematical Induction, Fibonacci Numbers, Limiting Ratio, Golden Ratio.

1. Introduction

The concept of determinants was extensively discussed by Cardano, Leibniz, Cramer, Jacobi, Vandermonde, Laplace and several other mathematicians. Since then, the study of determinants found its way in applications in various fields of mathematics. Today the study of determinants has been very basic and central part of higher mathematics. In this paper, we discuss evaluation of certain kinds of determinants of some particular form containing just three numbers 0, 1 and 3. We see that the value of such determinants turns out to be the most famous numbers in all of mathematics. We also prove the connection of given determinant with the Golden Ratio.

2. Description

The main focus of this paper is to evaluate the value of the following $n \times n$ determinant. D_n defined by

$$D_n = \begin{vmatrix} 3 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 3 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 3 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 3 & 1 & \cdots & 0 \; 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & 3 \; 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 \; 3 \end{vmatrix} \rightarrow (2.1)$$

From the definition of (2.1), we see that the main diagonal entries are all 3, entries below and above the main diagonal (sub-diagonal and super-diagonal) entries are 1 each and all other entries are 0.

3. Definition

The sequence of numbers defined by the recurrence relation $F_{n+2} = F_{n+1} + F_n \ (3.1), n \geq 0$ where $F_0 = 0, F_1 = 1$ are called Fibonacci Numbers. These numbers play a significant role in explaining beauty and applications of mathematics in almost all branches of Science and Technology.

Using the recurrence relation (2), the Fibonacci numbers F_n for $n \geq 0$ are given by 0,1,1,2,3,5,8,13,21,34,55,89,144,233, . . .

3.1 Golden Ratio

Golden Ratio denoted by φ is a real number given by $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618 \quad (3.2)$. This number is a positive real root of the quadratic equation $x^2 - x - 1 = 0$. The other root of this quadratic equation will be $\frac{1-\sqrt{5}}{2}$. In view of (3.2), $\frac{1-\sqrt{5}}{2} = -\frac{1}{\varphi} \quad (3.3)$. Also, $\varphi^2 = \varphi + 1 \quad (3.4)$

4. Theorem 1

The value of the determinant D_n is given by $D_n = F_{2n+2} \quad (4.1)$ where F_{2n+2} is the $(2n+2)$th Fibonacci number.

Proof: We prove this by mathematical induction on the order of the determinant *n*.

From (2.1), we notice that $D_1 = 3 = F_4 = F_{2(1)+2}$

Thus the result is true for *n* = 1. Hence by Induction Hypothesis, we assume that the result is true up to *n* = *k*. We prove it for *n* = *k* + 1. Evaluating the determinant for *n* = *k* + 1 we get

$$D_{k+1} = \begin{vmatrix} 3\,1\,0\,0\,0 \cdots & 0 \\ 1\,3\,1\,0\,0 \cdots & 0 \\ 0\,1\,3\,1\,0 \cdots & 0 \\ 0\;0\,1\,3\,1 \cdot\cdot & 0 \\ \cdots\cdots\cdots & \\ 0\,0\,0\,0\,0 \cdot\cdot\, 3 & 1 \\ 0\,0\,0\,0\,0 \cdot\cdot\, 1 & 3 \end{vmatrix} = 3\begin{vmatrix} 3\,1\,0\,0 \cdots & 0 \\ 1\,3\,1\,0 \cdots & 0 \\ 0\,1\,3\,1 \cdot\cdot & 0 \\ \cdots\cdots\cdots & \\ 0\,0\,0\,0 \cdot\cdot\, 3 & 1 \\ 0\,0\,0\,0 \cdot\cdot\, 1 & 3 \end{vmatrix} - 1 \times \begin{vmatrix} 1\,1\,0 \cdots & 0 \\ 0\,3\,1\,0 \cdots & 0 \\ 0\,1\,3\,1 \cdot\cdot & 0 \\ \cdots\cdots\cdots & \\ 0\,0\,0 \cdot\cdot\, 3 & 1 \\ 0\,0\,0 \cdot\cdot\, 1 & 3 \end{vmatrix} \quad (4.2)$$

Notice that the two determinants on the right hand side of (4.2) are of orders $k \times k$representing the minors of the non-zero entries 3, 1 in the first row of D_{k+1}.

The first determinant in the right hand side of (4.2). is the exact copy of D_{k+1},but it will be of order $k \times k$. Hence it should be D_k.

If we now try to evaluate the second determinant through first column, we observe that we get only one minor corresponding the first row-first column entry 1 which is again a copy of D_{k+1},but it will be of order $(k-1) \times (k-1)$. Hence it should be D_{k-1}.

Using these observations in (4.2), we get

$$D_{k+1} = 3D_k - D_{k-1} \quad (4.3)$$

But by Induction Hypothesis we know that the result is true up to *n* = *k*. Hence we get

$$D_k = F_{2k+2}, D_{k-1} = F_{2k}$$

Thus, equation (4.3) now becomes

$$D_{k+1} = 3F_{2k+2} - F_{2k}$$

Now using the recurrence relation of Fibonacci numbers as defined in (3.1), we get

$$D_{k+1} = 3F_{2k+2} - F_{2k} = 2F_{2k+2} + \left(F_{2k+2} - F_{2k}\right) = 2F_{2k+2} + F$$

$$= F_{2k+2} + F_{2k+3} = F_{2k+4}$$

Thus, we have $D_{k+1} = F_{2k+4}$. Thus the result is also true for *n* = *k* + 1. Hence by Induction Principle, the theorem must be true for all natural numbers *n*. This completes the proof.

5. Theorem 2

If F_{2n+1} is the $(2n+1)$th Fibonacci number, then

$$D_{n-1} \times D_n = F_{2n+1}^2 - 1 \quad (5.1)$$

Proof: By Cassini's identity of Fibonacci numbers, (see **[2]**) we know that

$$F_{r-1} \times F_{r+1} - F_r^2 = (-1)^r \quad (5.2)$$

Now using (4.1) of Theorem 1, we get

$$D_{n-1} \times D_n = F_{2n} \times F_{2n+2}$$

Considering $r = 2n + 1$ in (5.2), the previous equation becomes

$$D_{n-1} \times D_n = F_{2n} \times F_{2n+2} = (-1)^{2n+1} + F_{2n+1}^2 = F_{2n+1}^2 - 1$$

This completes the proof.

6. Theorem 3

If φ is the Golden Ratio then as $n \to \infty$, we have

$$\frac{D_{n+1}}{D_n} = \varphi + 1 \quad (6.1)$$

Proof: Using equation (4.1) of Theorem 1, we get

$$\frac{D_{n+1}}{D_n} = \frac{F_{2n+4}}{F_{2n+2}} = \frac{F_{2n+4}}{F_{2n+3}} \times \frac{F_{2n+3}}{F_{2n+2}} \quad (6.2)$$

We know that (see **[1]**) the ratio of consecutive Fibonacci numbers in the limiting case is precisely the Golden Ratio. That is, as $r \to \infty$, we have $\frac{F_{r+1}}{F_r} = \varphi$ (6.3)

Now using (3.4) and subsituting (6.3) in (6.2), as $n \to \infty$, we have

$$\frac{D_{n+1}}{D_n} = \varphi \times \varphi = \varphi^2 = \varphi + 1$$

This completes the proof.

7. Theorem 4

If φ is the Golden Ratio then as $n \to \infty$, we have $D_n = (\varphi + 1)^n$ (7.1)

Proof: Using (4.3) of theorem 1, for any natural number n, we have

$$D_{n+2} - 3D_{n+1} - D_n = 0 \qquad (7.2)$$

This recurrence relation leads to the auxiliary equation $m^2 - 3m + 1 = 0$. The roots of this quadratic equation are given by $m = \frac{3 \pm \sqrt{5}}{2}$

Among these two roots, in view of equations (3.2), (3.3) and (3.4), we notice that

$$\frac{3+\sqrt{5}}{2} = 1 + \frac{1+\sqrt{5}}{2} = 1 + \varphi, \frac{3-\sqrt{5}}{2} = 1 + \frac{1-\sqrt{5}}{2} = 1 - \frac{1}{\varphi} = \frac{\varphi - 1}{\varphi} = \frac{1}{\varphi+1}$$

Thus the solution of (7.2) is given by

$$D_n = (\varphi+1)^n + \frac{1}{(\varphi+1)^n} = \varphi^{2n} + \frac{1}{\varphi^{2n}} \qquad (7.3)$$

Now we notice that $0 < \frac{1}{\varphi+1} < 1$. Hence if we consider the limit as $n \to \infty$, then $\frac{1}{(\varphi+1)^n} \to 0$.

Thus as $n \to \infty$, from (7.3), we have $D_n = (\varphi + 1)^n$. This completes the proof.

8. Conclusion

Through four theorems dealt in this paper, we have established various properties of the given $n \times n$ determinant in (2.1). In particular, theorem 1, proves that fact that the given determinant D_n is precisely the ($2n$+2)th Fibonacci number. In theorem 2, we proved an interesting result concerning the product of two consecutive order determinants which turns to be square of (2n+1)th Fibonacci number minus 1. In theorem 3, we proved that the ratio of successive order determinants as n is very large is one added with Golden Ratio. Thus the determinant of order $n + 1$ is (φ + 1) times the determinant of order n when n is very large. Finally, in theorem 4, we proved that asymptotically the value of D_n is nth power of φ + 1. By considering a simple determinant using just three numbers 0, 1, 3 (which were themselves Fibonacci numbers) we proved variety of interesting results in this paper. We can try to extend the ideas provided in this paper to obtain further new results. Also, we can explore further by considering three numbers say from Lucas Sequence or some particular interesting integer sequence and discuss those properties as done in this paper. It is very surprising that the nth order determinant in the limiting case depends on the Golden Ratio, which at first sight is completely unobvious.

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