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On Some Stochastic Epidemic Models

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Abstract
In this article, we present some stochastic non-linear epidemic models related to Covid-19. It is very hard to get exact solutions for the non-linear such models, but we have successfully obtained exact solutions of some suitable non-linear stochastic cases. The general stochastic logistic model is solved; and some properties related to functions of Wiener process are studied.

Keywords: Epidemic non-linear stochastic models- Wiener processes- Exact stochastic solutions Stochastic logistic models.
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1. Introduction

Coronavirus (COVID-19) has continued to be a global threat to public health. As the matter of fact, it needs unreserved effort to monitor the prevalence of the virus. However, applying an effective prediction of the prevalence is thought to be the fundamental requirement to effectively control the spreading rate. The first case of coronavirus (COVID-19) outbreak was reported on December 31, 2019 in Wuhan, the central province of China, and then, it has been conveyed the global pandemic owning severe new type of threat to human health and life in a continues method. According to World Health Organization (2020), more than 15 million COVID-19 cases, 600,000 deaths, and 9 million recoveries have been currently reported.

When it comes to Africa, most of the cases were imported from European countries (Gilbert et al., 2020). Though the situation has been alleviated in China, it has been worthy increasing in Europe and America. It, recently, has also been worsening situation in Africa. Reportedly, there have been more than 750,000 confirmed cases, 15,000 deaths and 450,000 recoveries (Takele, 2020).

Particularly, South Africa, Egypt and Algeria are the most affected countries. Likewise, the pandemic has been reported as an exponentially spreading in East African countries such as Ethiopia, Djibouti, Somalia, Kenya and Sudan. There have been various factors that would facilitate the rapid spreading of the virus in those countries. To mention some, poor social-economic condition of the people, scarce medical supplies, poor medical conditions and low virus testing efficiency are the fundamental factors that would facilitate the rabid transmission of the pandemic.

This research aims to find a framework or approach which interpret the COVID-19 development. It introduces the Bayesian and time series models and how they could be useful for interpreting the evolution of COVID-19 and its patterns. Accordingly, the following objectives had been identified for the research:

1. Attempt a unified approach to the various models, which have appeared in the literature. This is of some value in indicating that certain features follow as consequences of a general model, which embraces many of the particular models.
2. Show the extent to which the different models provide satisfactory descriptions of observed phenomena.

Bayesian inference is a common method for conducting parameter estimation for dynamical systems. Despite the prevalent use of Bayesian inference for performing parameter estimation for dynamical systems, there is a need for a formalized and detailed methodology. This paper introduces solutions to the prediction of the disease in its different stages using the Bayesian, Maximum likelihood methods and time series models. The current paper is divided into several sections. The next section states the general stochastic logistic model. The third section states the stochastic susceptible- infective (SIS) model. The fourth section introduces the exact solution of fractional stochastic model.

2. General Stochastic Logistic Model

Let (Ω, F, F_t, P) be a filtered probability space and let {W(t): t ≥ 0) be a standard Wiener process adapted to the filtration $\{F_t,\ t \geq 0\}$.

Consider the following general stochastic logistic model

$dT(t) = f_1(t)T(t)[1 - f_2(t)T(t)]dt + f_3(t)T(t)dW$ (2.1).

With the following initial condition

$T(0) = T_o$.(2.2).

Where $T(t)$is the number of the infected cases at time at time$t$, $f_1(t)$is defined as the rate of infection at time t,f_2^{-1}(t) and f_3(t) is volatility function.

All the functions $f_i(t)$are measurable and bounded,f_i≥ 0, on [0, ∞], $i = 1,2,3$.

It is supposed that the initial value T_o is a random variable independent on $W(t)$such that $E(T_o^2)$ exists, where $E(X)$is the expectation of the random variable X.

Let us try to find the stochastic process $T(t)$.

Set $Y(t) = T^{-1}(t)$and using the chain rule of Itồ, we get

$dY(t) = [f_1(t)f_2(t) + \{f_3^2(t) - f_1(t)\}Y(t)]dt - f_3(t)Y(t)dW(t)$,

(2.3)

Set $Y(t) = Y_1(t)Y_2(t)$where

$dY_1(t) = \{f_3^2(t) - f_1(t)\}Y_1(t)dt - f_3(t)Y_1(t)dW(t), Y_1(0) = 1$,

$dY_2(t) = A(t)dt + B(t)dW(t),\ Y_2(0) = T_o^{-1}$.

The functions $A(t), B(t)$to be chosen.

Then

$dY(t) = Y_2(t)dY_1(t) + Y_1dY_2 - f_3(t)Y_1(t)B(t)dt = \{f_3^2(t) - f_1(t)\}Y(t)dt - f_3(t)Y(t)dW(t) + Y_1(t)[A(t)dt + B(t)dW(t)] - f_3(t)Y_1(t)B(t)dt$.

(2.4)

From (2.3) and (2.4), we get

$$Y_1(t)B(t) = 0 \ and \ Y_1(t)A(t) - f_3(t)B(t)Y_1(t) = f_1(t)f_2(t).$$

Consequently, $B(t) = 0 \ and \ A(t) = Y_1^{-1}(t)f_1(t)f_2(t)$.

Notice that;

$Y_1(t) = \exp\,[\int_0^t F(s)ds - \int_0^t f_3(s)dW(s)]$,

Where $F(t) = \frac{1}{2}f_3^2(t) - f_1(t)$.

We have Y_1 (t) > 0 almost surely, consequently;

$$Y_2(t) = T_o^{-1} + \int_0^t f_1(s)f_2(s)Y_1^{-1}(s)ds.$$

Also Y_2 (t) > 0 almost surely, thus the stochastic process $T(t)$is given by;

$T(t) = [Y_1(t)Y_2(t)]^{-1}$,

See [1-6]

If $f_i(t) = a_i, i = 1,2,3$ are constants, we get

$Y_1(t) = \exp\,[\left\{\frac{1}{2}a_3^2 - a_1\right\}t - a_3W(t)]$,

$$Y_2(t) = T_o^{-1} + a_1a_2\int_0^t \exp\left[\left\{a_1 - \frac{1}{2}a_3^2\right\}s + a_3W(s)\right]ds,$$

$$T(t) = \frac{T_o exp\,[\{a_1 - {}^1\!/_2\, a_3^2\}t + a_3W(t)}{1 + a_1a_2T_o\int_0^t \exp\,[\{a_1 - {}^1\!/_2\, a_3^2\}s + a_3W(s)ds]}$$

3. Stochastic- Susceptible- Infective (SIS) Model

The following stochastic model is considered

$$dS(t) = [-rS(t)I(t) + \alpha I(t)]dt + \sigma(t)S(t)dW(t)$$

$$dI(t) = [rS(t)I(t) - \alpha I(t)]dt + \sigma(t)I(t)dW(t),$$

(3.1)

Subject to, initial conditions

$$S(0) = S_o,\quad I(0) = I_o$$

r> 0, S_0> 0, I_0> 0,$S_o \ and \ I_o$ are deterministc, r is the infectivity coefficient of the typical Lotka-volterra interaction term, and α is the recovery coefficient, we have;

$$S(t) + I(t) = \sigma(t)[S(t) + I(t)]dW(t)$$

Thus,

$$S(t) + I(t) = (S_o + I_o)\exp\,[-\frac{1}{2}\int_0^t \sigma^2(s)ds + \int_0^t \sigma(s)dW(s)]$$

If σ (t)= σ is a constant, we get

$$S(t) + I(t) = (S_o + I_o)\exp\left[-\frac{1}{2}\sigma^2t + \sigma W(t)\right].$$

Since $E[\exp\{\sigma W(t)\}] = \exp\left[\frac{1}{2}\sigma^2 t\right]$, it follows that $E[S(t) + I(t)] = S_o + I_o$.

We can write:

$dI(t) = rI(t)\left[(S_o + I_o)\exp\{-\frac{1}{2}\sigma^2 t + \sigma W(t)\right\} - I(t)]dt - \alpha I(t)dt + \sigma I(t)dW(t).$
(3.2)

The last equation is a stochastic logistic equation similar to (2.1).Equation (3.2) can be written in form

$dI(t) = f_1(t)I(t)[1 - f_2(t)I(t)] + \sigma I(t)dW(t)$
(3.3)

where;

$$f_1(t) = r(S_o + I_o)\exp\left[-\frac{1}{2}\sigma^2 t + \sigma W(t)\right] - \alpha,\ f_2(t) = f_1^{-1}(t)$$

Thus the stochastic process I (t) is given by;

$$I(t) = \frac{I_o \exp\left[-{}^1\!/_2\,\sigma^2 t + \sigma W(t) + \int_0^t f_1(s)ds\right]}{1 + I_o\int_0^t \exp\left[-{}^1\!/_2\,\sigma^2 s + W(s) + \int_0^t f(\theta)d\theta\right]ds}$$

Consequently the stochastic process S (t) is given by;

$$S(t) = (S_o + I_o)\exp\left[-\frac{1}{2}\sigma^2 t + W(t)\right] - I(t).$$

See [8-11].

4.Exact Solution of Fractional Stochastic Model

We shall find in this section the exact solution of a fractional stochastic model.

Let us first find, exact solution of the following fractional logistic model:

$$\frac{d^\alpha u(t)}{dt^\alpha} = Ku(t)[1 - f(t)u(t)]$$

With the initial condition

$$u(0) = u_o$$

Where$f$ is a measurable bounded function on $[0, \infty]$.

Notice that at$\alpha = 1$, we have;

$\frac{dv}{dt} = -Kv(t) + Kf(t),$
(4.1)

Where

$v(t) = u^{-1}(t)$.(4.2)

Consequently;

$u(t) = \frac{e^{Kt}u_o}{1 + Ku_o\int_0^t f(\theta)e^{K\theta}d}$
(4.3)

We can generalize the results in [2]

Consider the fractional version of equation (4.1).

$$\frac{d^\alpha v(t)}{dt^\alpha} = -Kv(t) + Kf(t), 0 < \alpha \le 1.$$

According to the results in [] we can write

$$v(t) = \int_0^\infty \zeta_\alpha(\theta)e^{-Kt^\alpha\theta}u_0^{-1}d\theta + \alpha\int_0^t\int_0^\infty \theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta)e^{-K(t-\eta)^\alpha\theta}f(\eta)d\theta d\eta$$

Where $\zeta_\alpha(\theta)$is the stable probability density function defined on $(0, \infty)$

So,

$$v(t) = u_o^{-1}E_\alpha(-Kt^\alpha) + \int_0^t f(\eta)\frac{d}{d\eta}E_\alpha(-K(t-\eta)^\alpha)d\eta$$

Where E_α (t) is the Mittaglefler function.

For f (t) =1 we get;

$u(t) = u_o[(1 - u_o)E_\alpha(-Kt^\alpha) + u_o]^{-1}$,

Which is the formula in [1]

At $\alpha = 1$ we get formula (4.3).

Consider now the following fractional stochastic logistic model;

$$u(t) = u(0) + \frac{K}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[u(s) - f(s)u^2(s)ds + \frac{\sigma}{\Gamma(\alpha)}\int_0^t (t - s^{\alpha-1}u(s)dW(s).$$

At$\alpha = 1$, we have the following equation

$$u(t) = u(0) + K\int_0^t [u(s) - f(s)u^2(s)]ds + \sigma\int_0^t u(s)dW(s)$$

Set $Y(t) = u^{-1}(t)$, we get

$$dY(t) = Kf(t)dt + (\sigma^2 - K)Y(t)dt - \sigma Y(t)dW(t).$$

The fraction version of last equation is given by;

$$V(t) = V(0) + \frac{K}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[f(s) + (\sigma^2 - K)V(s)]ds - \frac{\sigma}{\Gamma(\alpha)}\int_0^t (t - s)^{\alpha-1}V(s)dW(s).$$

Thus, according to [3] we get;

$V(t) = \int_0^\infty \zeta_\alpha(\theta)e^{-(K-\sigma^2)t^\alpha\theta}V(0)d\theta + \int_0^t\int_0^\infty \alpha\theta(t - \eta)^{\alpha-1}K\zeta_\alpha(\theta)e^{-(K-\sigma^2)(t-\eta)^\alpha\theta}f(\eta)d\eta - \sigma\int_0^t\int_0^\infty \alpha\theta(t - \eta)^{\alpha-1}\zeta_\alpha(\theta)e^{-(K-\sigma^2)(t-\eta)^\alpha\theta}V(\eta)dW(\eta).$

The existence and uniqueness of the stochastic process can be proved by the method of successive approximations , see [7-17]

5.Conclusion

In the present paper we have considered some stochastic epidemic models. Exact stochastic solutions are obtained for general non-linear systems. The fractional stochastic logistic equation is solved.

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