Journal of Positive School Psychology http://journalppw.com
2022, Vol.6, No.4, 2124-2134

Numerical methods for solving integro partial differential equation with fractional order

Mahmoud. M. Elborai*,[1], M. A. Abdou[2], W. El-Sayed[3], Sh. I. Awed[4]
[1,3] *Department of Mathematics and computer sciences, Faculty of Science, Alexandria University*
[2,4] *Department of Mathematics, Faculty of Education, Alexandria University*

[1] m.m.elborai@alexu.edu.eg
1.m_m_elborai@yahoo.com
[1] abdella_777@yahoo.com
[1] wagdygoma@alexu.edu.eg
[1] Shenoudaissa@alexu.edu.eg

Abstract

Some fractional order integro partial differential equations are investigated. Some initial value problems are considered using the Picard technique. Furthermore, the topic of stability is discussed. In addition, the Cauchy problem is transformed to a nonlinear algebraic system. Finally, numerical results are produced for different values of fractional order α *and time* t.

Keywords: Adomian Decomposition Method; Integral equation; Partial differential equation.

1 Introduction

The Cauchy problems have many applications in different sciences especially in mathematical physics , theory of elasticity , mathematical engineering and contact problems. See [1, 2, 3, 4, 5]. For a linear fractional evolution, the author in [6, 7] , discussed the solution of Cauchy problem with singular kernel in a Banach space, Also , in terms of several probability density function. The same author obtains the general solution. The same authors in [9, 28, 29] solved the nonlinear fractional differential equation using semigroup method. In [8], the authors using an analytic and numerical methods to discuss the solutions of some linear partial differential equations. In [8], the authors employedProduct Nystrom method to get numerical solutions in $L_2[a,b] \times [0,T), T < 1$.

Consider the following integro partial differential equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + \mathcal{F}(x,t,\Im u), \quad 0 < \alpha \leq 1, \qquad u(x,0) = \phi(x), \tag{1.1}$$

where $\Im$ is an integral operator which defined by :

$$\Im u = \int_0^t k(x,\theta) u(x,\theta)\, d\theta, \tag{1.2}$$

where $\|k(x,t)\| \leq K$, K is a constant, and $u(x,t)$ is real valued function on $\mathbb{R} \times [0,T]$, $T < 1$. and $\Im u(x,t)$ is a linear closed and bounded integral operator defined on any subset of $\mathbb{R} \times [0,T]$.

2 Basics and fundamentals

In the first case, we will discuss the existence of the solution. For this aim, consider the following Cauchy problem

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + \mathcal{F}(x,t,\Im u), \quad 0 < \alpha < 1 \qquad u(x,0) = \phi(x) \tag{2.1}$$

Let us assume that the given function $\mathcal{F}$ satisfies Holder condition:

(a) $\|\mathcal{F}(x,t_2,\omega) - \mathcal{F}(x,t_1,\omega)\| \leq l(t_2 - t_1)^\beta, \quad t \in [0,T],$ (2.2)

for all $t_2 > t_1, t_1, t_2 \in [0,T]$ and all $x, \omega \in \mathbb{R}$, where l and β are positive constants, $\beta \leq 1$,

(b) $\mathcal{F}$ satisfies the Lipschitzcondition

$$\|\mathcal{F}(x,t,\omega) - \mathcal{F}(x,t,\omega^*)\| \leq l_1|\omega - \omega^*|, \tag{2.3}$$

for all $x, \omega, \omega^* \in \mathbb{R}$ and all $t \in [0,T]$,where $l_1 > 0$.

(c) $\|\mathcal{F}(x,t,\omega)\| \leq l^*|\omega|, (l^* > 0)$ (2.4)

($\mathcal{F}$ is bounded by the third argument.)

where $t_1 > 0, t_2 \in [0,T], h \in \mathbb{R}$ and k_2 is a positive constant.

The integral convolution between f and the generalized function,lies at the heart of fractional calculus theory. See Gelfand and Shilov on [9, 10, 11], therefore, the integral of order $\alpha > 0$ is defined as follow:

$$I^\alpha[f(t)] = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} f(\theta)\, d\theta. \tag{2.5}$$

If $0 < \alpha < 1$, the fractional derivative of order $\alpha$is given as

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\theta)}{(t-\theta)^\alpha}\, d\theta, \tag{2.6}$$

where f is given function.

Now, it is possible to reframe the Cauchy problem (2. 1) in this way,

$$u(x,t) = \phi(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} \frac{\partial^2 u(x,t)}{\partial x^2} d\theta + \int_0^t (t-\theta)^{\alpha-1} \mathcal{F}(x,\theta,\Im u(x,\theta))\, d\theta \tag{2.7}$$

where $\mathbb{C}(\mathbb{R} \times [0,T])$ denotes all continuous functions u set on $\mathbb{R} \times [0,T]$, and define on $\mathbb{C}(\mathbb{R} \times [0,T])$ a norm by :

$$\|u(x,t)\| = \max_{x,t} \|u(x,t)\|_{\mathbb{R}}, \quad \forall t \in [0,T], x \in \mathbb{R}. \tag{2.8}$$

For $\lambda > 1$and$0 < \delta < 1$, it is clear to see that

$$\int_0^t (t-\eta)^{\delta-1} d\eta \leq \left(\frac{1}{\lambda}\right)^{\delta-1} t, \tag{2.9}$$

and

$$\int_0^t e^{\lambda\eta}(t-\eta)^{\delta-1} d\eta \leq \left(\frac{1}{\lambda}\right)^\delta \left[1 + \frac{1}{\delta}\right] e^{\lambda t}. \tag{2.10}$$

Consider the following integral equation, (See [6,7])

$$u(x,t) = \int_0^\infty \xi_\alpha(\theta) \exp\left(-\frac{(x-y)^2}{4t^\alpha\theta}\right)\phi(x)\, d\theta + \alpha \int_0^t \int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta) \exp\left(-\frac{(x-y)^2}{4(t-\eta)^\alpha\theta}\right)\omega(x,\eta) \tag{2.11}$$

where $\xi_\alpha(\theta)$ is a probability density function, which defined on $[0,\infty)$, see [6]. Also,

$$\omega(x,t) = \mathcal{F}(x,t,\Im u(x,t)) = \mathcal{F}(x,t,W(x,t)), \tag{2.12}$$

and

$$W(x,t) = \int_0^{t_1}\int_0^\infty \xi_\alpha(\theta)k(x,\theta) \exp\left(-\frac{(x-y)^2}{4t^\alpha\theta}\right)\phi(+ \alpha \int_0^{t_1}\int_0^t \int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)k(x,\theta) \exp\left(-\frac{(x-y)^2}{4(t-\eta)^\alpha\theta}\right)\omega \tag{2.13}$$

3 The existence and uniqueness solution

Theorem 3.1

The solution of equation (2.9)exists in the space $\mathbb{C}(\mathbb{R}\times[0,T])$.

Proof.

Using the method of successive approximations, let

$$\omega_{k+1}(x,t) = \mathcal{F}(x,t,\Im u_k), \quad k = 0,1,2,\ldots \tag{3.1}$$

We have from Lipchitz condition and **Error! Reference source not found.** the following :

$$\|\omega_{k+1} - \omega_k\| \le l_1 \max_{x,t}\|\Im u_k - \Im u_{k-1}\| \le M \max_{x,t}\|u_k(x,t) - u_{k-1}(\tag{3.2}$$

where l_1 is constant and $M = (B+KT)l_1$.

Since,

$$\|u_k(x,t) - u_{k-1}(x,t)\| \le \int_0^t\int_0^\infty\int_{-\infty}^\infty \frac{\alpha\theta(t-\eta)^{\alpha-1}}{\sqrt{4\pi(t-\eta)^\alpha\theta}}\xi_\alpha(\theta)\|\omega_k(x,t) - \omega_{k-1}(x,t)\| \exp\left(-\frac{(x-y)^2}{4(t-\eta)^\alpha\theta}\right) dy\, d\eta\, d \tag{3.3}$$

From **Error! Reference source not found.**and **Error! Reference source not found.**, we have

$$\|\omega_{k+1}(x,t) - \omega_k\| \le \int_0^t\int_0^\infty\int_{-\infty}^\infty \frac{\alpha\theta(t-\eta)^{\alpha-1}}{\sqrt{4\pi(t-\eta)^\alpha\theta}}\xi_\alpha(\theta) \exp\left(-\frac{(}{4(}\right. - \omega_{k-1}(x,t)\|\, dy\, d\eta\, d\theta, \tag{3.4}$$

In (3.5), we used the following substitutions:

$$\begin{cases} \nu = \alpha(1-\gamma), \\ M^* = W\alpha\int_0^\infty \theta^{\alpha-1}\xi_\alpha(\theta)d\theta, \\ \int_0^\infty \frac{1}{\sqrt{4\pi(t-\eta)^\alpha\theta}} \exp\left(-\frac{(x-y)^2}{4(t-\eta)^\alpha\theta}\right)dy \le \end{cases}$$

Finally, using (2.11), (2.12) one gets

$$e^{-\lambda(x+t)}\max_{x,t}\|\omega_{k+1}(x,t) - \omega_k\| \le M^*\left(\frac{1}{\lambda}\right)^\nu\left[1 + \frac{1}{\nu}\right]\max_{x,t}\left[e^{-\lambda(x+t)}\|\omega_k - \omega_{k-1}\|\right], \tag{3.5}$$

In **Error! Reference source not found.**, we can choose λ sufficiently large such that

$$M^*\left(\frac{1}{\lambda}\right)^\nu\left[1+\frac{1}{\nu}\right] = \mu <$$

By induction,we can obtain

$$e^{-\lambda(x+t)}\max_{x,t}\|\omega_{k+1}(x,t) - \omega_k\| \le (\mu)^n \max_{x,t}\left[e^{-\lambda(x+t)}\|\omega_1 - \omega_0\|\right]$$

where $\omega_0(x,t)$ can be take the zero function. Thus the sequence $\{\omega_n(x,t)\}$ converges uniformly in the space $\mathbb{C}(\mathbb{R}\times[0,T])$ to a continuous function $\mathcal{F}(x,t,\Im u)$, which satisfies

$$\int_0^\infty \xi_\alpha(\theta)\exp\left(-\frac{(x-y)^2}{4t^\alpha\theta}\right)\phi(x)d\theta + \alpha\int_0^t\int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)\exp\left(-\frac{(x-y)^2}{4(t-\eta)^\alpha\theta}\right)\mathcal{F}(x,t,\Im \tag{3.6}$$

Hence, the series $\sum_{k=0}^\infty\|\omega_{n+1} - \omega_n\|$ converges uniformly in $\mathbb{R}\times[0,T]$ where $\xi_\alpha(\theta)$ is the probability density function defined on $[0,\infty)$. Therefore, the solution is obtained.

Theorem 3.2 *The solution of equation (2.9) is unique.*

Proof. Rewrite the partial operators in the form:

$$\frac{\partial^\alpha u_i(x,t)}{\partial t^\alpha} - \frac{\partial^2 u_i(x,t)}{\partial x^2} = \omega_i(x,t) \quad ,i = 1,2. \tag{3.7}$$

Where

$$u_i(x,t) = \int_0^\infty \int_{-\infty}^\infty \frac{1}{\sqrt{4\pi t^\alpha \theta}} \xi_\alpha(\theta) \exp\left(-\frac{(x-y)^2}{4t^\alpha \theta}\right) \phi(y)\, dy\, d\theta + \int_0^t \int_0^\infty \int_{-\infty}^\infty \frac{\alpha\theta (t-\eta)^{\alpha-1}}{\sqrt{4\pi (t-\eta)^\alpha \theta}} \xi_\alpha(\theta) \exp\left(-\frac{(x-y)^2}{4(t-\eta)^\alpha \theta}\right) \omega_i(y,\eta)\, dy\, d\eta\, d\theta, \quad (3.8)$$

and

$$\omega_i = \mathcal{F}(x,t,\Im u_i), i = 1,2. \quad (3.9)$$

Since $\hat{o}(x,t,\Im u_i)$ satisfy Lipschitz condition on the third argument ,we get

$$\max_{x,t} ||\omega_1 - \omega_2|| \le l_1 \max_{x,t} ||\Im u_1 - \Im u_2|| \quad (3.10)$$

It is noticed that the integral operator given in (1. 2) has kernel function which has more than one possibility , it maybe continuous or discontinuous kernel ,therefore we prove the uniqueness of each case of them after the following statement:

$$||u_1 - u_2|| \le M \int_0^t (t-\eta)^{\alpha-1} ||\omega_1(y,\eta) - \omega_2(y,\eta)|| d\eta \quad (3.11)$$

Now, From (3.10) ,(3.11) we get

$$||u_1 - u_2|| \le M l_1 T \int_0^t (t-\eta)^{\alpha-1} ||K(x,\theta)|| ||u_1(y,\eta) - u_2(y,\eta)||\, d\eta, \quad (3.12)$$

In the case of continuous kernel we have $||K(x,t)|| \le K$ where K is a positive constant. We obtain

$$||u_1 - u_2|| \le W \int_0^t (t-\eta)^{\alpha-1} ||u_1(y,\eta) - u_2(y,\eta)||\, d\eta \quad (3.13)$$

where $W = M l_1 K$. Then

$$\max_{x,t} ||u_1 - u_2|| \le W\rho \int_0^t (t-\eta)^{\alpha-1} e^{\lambda(x+\eta)} dt \quad (3.14)$$

Thus

$$\max_{x,t} ||u_1 - u_2|| \le \mu\rho e^{\lambda(x+t)} \quad (3.15)$$

where $\mu = W\left(\frac{1}{\lambda}\right)^\alpha \left[1 + \frac{1}{\alpha}\right], \rho = \max_{x,t}\left[e^{-\lambda(x+t)} ||u_1 - u_2||\right]$.

By choosing λ is sufficiently large ,we deduce that $\mu < 1$,then

$$\rho \le \mu\rho \Rightarrow u_1 = u_2$$

which is the required.

In the case of discontinuous kernel we have

$$||K(x,t)|| \le \frac{K^*}{(t-\eta)^\gamma} \quad (3.16)$$

where K^* is a positive constant. In (3.12) we get directly :

$$||u_1 - u_2|| \le W \int_0^t (t-\eta)^{\alpha-1} \cdot \frac{1}{(t-\eta)^{\alpha\gamma}} = W \int_0^t (t-\eta)^{\delta-1} ||u_1(y,\eta) - \quad (3.17)$$

where $W = M l_1 K^*$ and $\delta = (1-\gamma)\alpha$. By the same manner of proof in the first case we can prove the second case which get us the required which is the solution is unique in the case of continuous kernel or discontinuous.

4 Stability of the solution

Theorem 4.1 *Suppose that $\{u_n\}$ is a sequence of solutions of equation (2.9) with the initial condition $u_n(x,0) = \phi_n$, where ϕ_n is a given real valued function , $(n = 1,2,3,\ldots)$. If $\left\{\frac{\partial^2 \phi_n}{\partial x^2}\right\}$ and $\left\{\int_0^\infty k(x,\theta)\phi_n(x)dx\right\}$ are uniformly convergence on $R \times [0,T]$. Then, $u_n(x,t)$ converges uniformly on $R \times [0,T]$ to a limit function $u(x,t)$ which is the solution of (2.9).*

Proof.

Let $\{u_n\}$ and $\{\omega_n\}$ be two sequence satisfies the following:

$$\frac{\partial^\alpha u_n^*}{\partial t^\alpha} - \frac{\partial^2 u_n^*(x,t)}{\partial x^2} = \omega_n(x,t), \quad n = 1,2,3,\ldots \quad (4.1)$$

where

$$u_n^* = u_n(x,t) - \phi_n(x)$$

then,

$$u_n^* = \int_0^t \int_0^\infty \theta(t\eta)^{\alpha-1} \xi_\alpha(\theta) e^{-\frac{(x-y)^2}{(t-\eta)^\alpha \theta}} \omega_n(x,\eta)\, d\theta\, d\eta$$

and

$$\omega_n(x,t) = \mathcal{F}(x,t,\Im u_n^* + \Im\phi_n(x)$$

Now,

$$\left\|\omega_n(x,t)-\omega_m(x,t)\right\| \leq \left\|\frac{d^2\phi_n(x)}{dx^2}-\frac{d^2\phi_m(x)}{dx^2}\right\| + \left\|\mathcal{F}(x,t,\Im(u_n^*(x,t)) + \Im(\phi_n(x))) - \mathcal{F}\big(x,t,\Im\big(u_m^*(x,t)\big) - \Im(\phi_m(x))\big)\right\|$$

In view of conditions (2.6) and the choosing the constants M, and $\forall\varepsilon>0, \exists\, N=N(\varepsilon)$, such that

$$\left\|\omega_n-\omega_m\right\| \leq (1+\mu^*)\varepsilon + M\int_0^t (t-\eta)^{\delta-1}\left\|\omega_n(y,\eta)-\omega_m(y,\eta)\right\| d\eta$$

such that $\forall\varepsilon>0, \exists\, N=N(\varepsilon)$,then

$$\left\|\omega_n-\omega_m\right\| \leq (1+\mu^*)\varepsilon + M\int_0^t (t-\eta)^{\delta-1}\left\|\omega_n(y,\eta)-\omega_m(y,\eta)\right\| d\eta, \quad \forall m,n\geq N$$
$$\leq (1+\mu^*)\varepsilon + M\rho_1\mu e^{\lambda(x+t)}$$

where $\mu = M\left(\frac{1}{\lambda}\right)^{\delta}\left[1+\frac{1}{\delta}\right], \rho_1 = \max_{x,t}\left[e^{-\lambda(x+t)}\left\|\omega_n-\omega_m\right\|\right]$

Then, $(1-\mu)\rho_1 \leq (1+\mu^*)\varepsilon e^{-\lambda(x+t)}$

For choosing λ sufficiently large , we get $\rho_1 \leq \varepsilon$

Since the space $\mathbb{R}\times[0,T]$ is complete space .Then the sequence $\{\omega_n\}$ converges uniformly on $\mathbb{R}\times[0,T]$ to a continuous function ω.So, the sequence $\{u_n^{å}\}$ uniformly converges to a continuous function u^*. Which is the required.

5 Numerical Results

The Adomian Decomposition Method (ADM) will be present in this section which represents a suitable method for obtaining analysis. This method was introduced by G.Adomain [13, 14, 15]. ADM is considered as one of the high-accuracy and fast computational methods for finding the solution of linear and non-linear equations, ordinary, partial, integral, fractional partial integro-differential equations [16, 17, 18, 19, 20, 21]. It's also been researched and used in a variety of fields, including viscoelastic fluid , biomedical , physics and other applications to see it in [22, 23, 24, 13, 25]. One of the advantages of this method is that it allows you to solve equations without being transformed into simpler versions. This approach [19] does not involve linearization, perturbation, discretization, or any unreasonable assumptions. A nonlinear operator is decomposed into a series of functions using the 1method. Each term in the series is derived from a polynomial derived from an analytic function's power series expansion. The Adomian approach is fairly basic in its abstract form.

In the following we introduce the algorithm of ADM, [26] , [27] by considering the following PDE in its general form:

$$L[u(x,t)] + R[u(x,t)] + N[u(x,t)] = G(x,t) \qquad (5.1)$$

where $L[.]$ is the partial derivative of the function u with respect to t which is a linear operator also its has an inverse $L^{-1}[.]$ which is integral operator, $R[.]$ is reminder linear operator, $N[.]$ is a nonlinear operator and $G(x,t)$ is given function (independent of u). By applying the inversion operator (L^{-1}) to both sides of (5.1) , we get

$$u(x,t) = \phi(x) + L^{-1}[G(x,t)] - L^{-1}[R[u(x,t)]] - L^{-1}[R[u(x,t)]] \qquad (5.2)$$

where $\phi(x)$ is a given function from the initial condition of the origin problem. The ADM postulate the unknown function $u(x,t)$ write in the series form :

$$u(x,t) = \textstyle\sum_{n=0}^{\infty} u_n(x,t$$

and in the case of nonlinear operator $N[u(x,t)]$ arise, we will substitute for N by:

$$N(u) = \sum_{n=0}^{\infty} A_n \qquad (5.3)$$

where A_n's are called the Adomian polynomials [13] is given by

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\delta^n}\left\{N\left(\sum_{i=0}^{n}\delta^i u_i(x,t)\right)\right\}\right]_{\delta=0} \qquad (5.4)$$

This case (nonlinear case)is arise in the most time when we use ADM [26]. We deal here with the linear case which not lead us to using N. To find the components $u_0, u_1, u_2, u_3, ...$ We use the following recurrence relations:

$$u_0(x,t) = \phi(x) + L^{-1}[G(x,t)] \qquad (5.5)$$

$$u_{k+1}(x,t) = -L^{-1}[R[u(x,t)]] - L^{-1}[R[u(x,t)]] \qquad (5.6)$$

Consequently, an approximate solution of problem (5.1) with initial condition $u(x,0) = \phi(x)$ is given by:

$$u_n(x,t); \sum_{k=0}^{n} u_k(x,t), \quad n \in \mathrm{N} \tag{5.7}$$

and, then

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) \tag{5.8}$$

The error estimate of the method can be established in the form :

$$R_n = |u(x,t) - u_n(x,t)| \tag{5.9}$$

Now we'll use the ADM algorithm to the following two problems and see how the numerical (computational) results change as the value of α, t changes. By taking $L[.] = \frac{\partial^\alpha}{\partial t^\alpha}[.]$ with its inverse $L^{-1}[.] = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\theta)^{\alpha-1}[.]d\theta$, $R[.] = \frac{\partial^2}{\partial x^2}[.]$ and the remainder part is $G(x,t) = \hat{o}(x,t,\Im u)$(we will remove u from here). The fractional calculus relations. also is used.

Example 5.1 *Consider the following fractional integro partial differential equation :*

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + \int_0^t x^3\theta^2 u(x,\theta)d\theta + g(x,t), \quad 0 < \alpha < 1 \tag{5.10}$$

with the exact solution :

$$u(x,t) = x^2t^2, \tag{5.11}$$

therefore after integrating (5.10) with respect to t ,we have

$$u(x,t) = x + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\theta)^{\alpha-1}\frac{\partial^2 u(x,\theta)}{\partial x^2}d\theta + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\theta)^{\alpha-1}\Im u(x,t)d\theta + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\theta)^{\alpha-1}g(x,t)d\theta, \tag{5.12}$$

To apply ADM ,we determine the components $u_0, u_1, u_2, \ldots$, from the relations

$$u_0(x,t) = \phi(x) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\theta)^{\alpha-1}g(x,t)d\theta,$$

$$u_k(x,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\theta)^{\alpha-1}\frac{\partial^2 u_{k-1}(x,\theta)}{\partial x^2}d\theta + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\theta)^{\alpha-1}I u_{k-1}d\theta\,, \quad k = 1,2,3,\ldots \tag{5.13}$$

where $\Im u$ is given by:

$$\Im u = \int_0^t k(x,\theta)u(x,\theta)d\theta \tag{5.14}$$

Hence, we will have the approximate solution from the relation (5.8) and the error estimate of the method can be established in the form (5.9). The end results of exact, approximate solutions, and errors for example(5.1) are now obtainable at $\alpha = 0.95$ for $t = 0.01, 0.1, 0.4, 0.9$ in the following table and figures, also we show the figure (2) where $\alpha = 0.5, t = 0.4$ and figure (3) $\alpha = 0.3, t = 0.3$, finally we present the 3D-figure (4) of exact, approximate solutions at $\alpha = 0.8$.

Figures

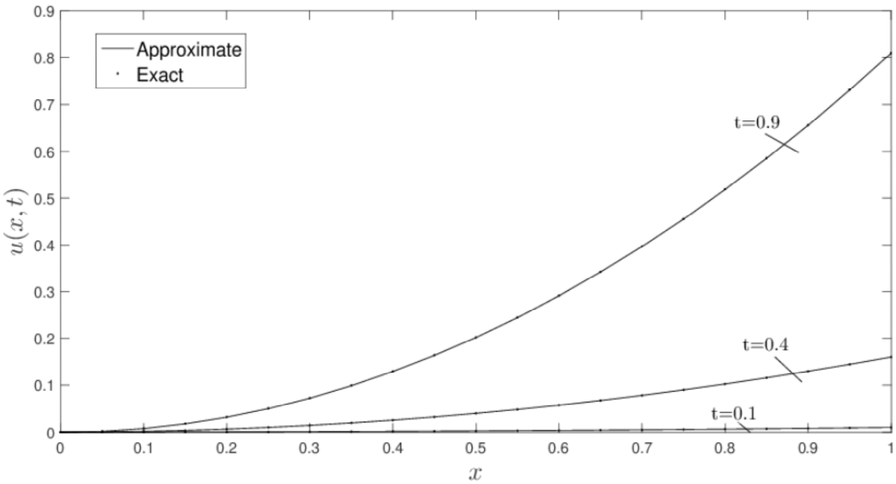

Figure 1: Represents the exact and the approximate solutions with different values of t at $\alpha = 0.95$.

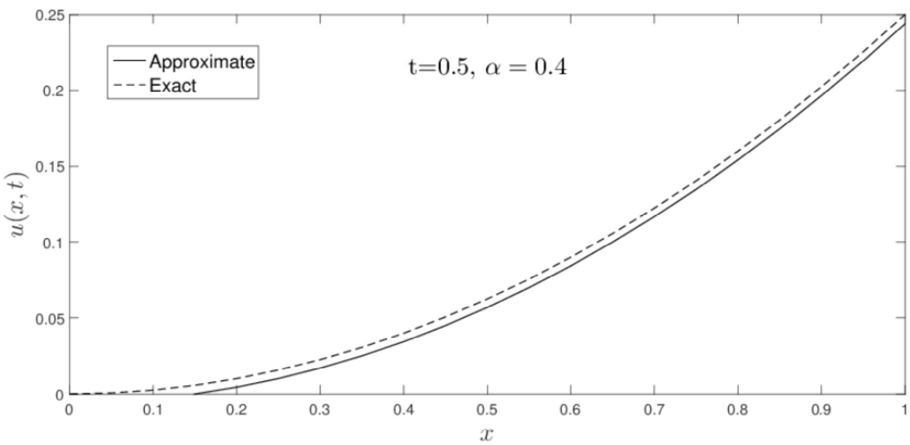

Figure 2: Variation of the exact and approximate solutions for $\alpha = 0.5, t = 0.4$.

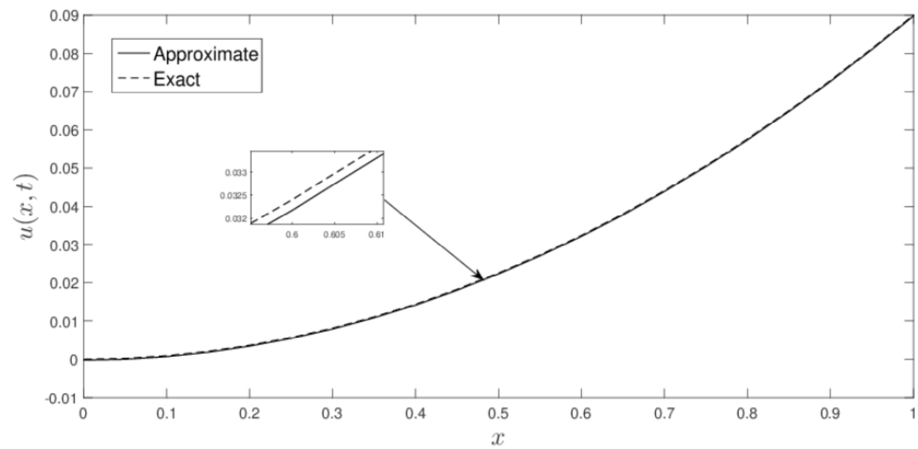

Figure 3: The behavior of the exact and approximate solutions for $\alpha = 0.3, t = 0.3$.

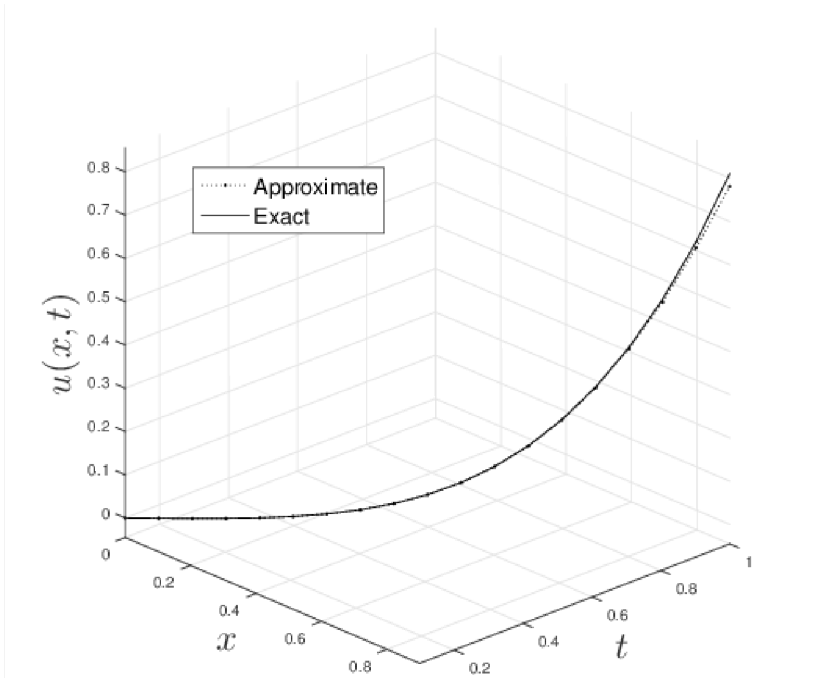

Figure 4: The behavior of the exact and approximate solutions for $\alpha = 0.8$.

0.1	
Approximate	Error
0.0001	1.44482×10^{-5}
0.0016	1.44483×10^{-5}
0.0049	1.44508×10^{-5}
0.01	1.44637×10^{-5}
0.0001	1.14×10^{-5}
0.0016	1.14×10^{-5}
0.0049	1.14×10^{-5}
0.01	1.14×10^{-5}
0.0001	1.05558×10^{-6}
0.0016	1.05559×10^{-6}
0.0049	1.05575×10^{-6}
0.01	1.05658×10^{-6}
0.0001	4.75156×10^{-9}
0.0016	4.75159×10^{-9}
0.0049	4.75216×10^{-9}
0.01	4.75515×10^{-9}

□	□	□ = 0.001			$t = 0.01$			
		Exact	Approximate	Error	Exact	Approximate	Error	Exact
0.001	0.10	1.0×10^{-8}	1.0×10^{-8}	1.07699×10^{-21}	1.0×10^{-6}	1.0×10^{-6}	1.42499×10^{-13}	8.56×10^{-5}
	0.40	1.6×10^{-7}	1.6×10^{-7}	8.73503×10^{-22}	0.000016	0.000016	1.42499×10^{-13}	0.001586
	0.70	4.9×10^{-7}	4.9×10^{-7}	7.41154×10^{-22}	0.000049	0.000049	1.42499×10^{-13}	0.004886
	1.00	1.0×10^{-6}	1.0×10^{-6}	6.35275×10^{-22}	0.0001	0.0001	1.42499×10^{-13}	0.009986
0.01	0.10	1.0×10^{-8}	1.0×10^{-8}	1.83469×10^{-21}	1.0×10^{-6}	1.0×10^{-6}	9.9368×10^{-14}	8.86×10^{-5}
	0.40	1.6×10^{-7}	1.6×10^{-7}	1.64113×10^{-21}	0.000016	0.000016	9.9368×10^{-14}	0.001589
	0.70	4.9×10^{-7}	4.9×10^{-7}	1.69407×10^{-21}	0.000049	0.000049	9.9368×10^{-14}	0.004889
	1.00	1.0×10^{-6}	1.0×10^{-6}	1.48231×10^{-21}	0.0001	0.0001	9.93681×10^{-14}	0.009989
0.1	0.10	1.0×10^{-8}	1.0×10^{-8}	6.30312×10^{-22}	1.0×10^{-6}	1.0×10^{-6}	2.65156×10^{-15}	9.89×10^{-5}
	0.40	1.6×10^{-7}	1.6×10^{-7}	6.08805×10^{-22}	0.000016	0.000016	2.65158×10^{-15}	0.001599
	0.70	4.9×10^{-7}	4.9×10^{-7}	5.29396×10^{-22}	0.000049	0.000049	2.65156×10^{-15}	0.004899
	1.00	1.0×10^{-6}	1.0×10^{-6}	4.23516×10^{-22}	0.0001	0.0001	2.65154×10^{-15}	0.009999
0.3	0.10	1.0×10^{-8}	1.0×10^{-8}	2.31611×10^{-22}	1.0×10^{-6}	1.0×10^{-6}	7.29295×10^{-19}	1.0×10^{-4}
	0.40	1.6×10^{-7}	1.6×10^{-7}	1.32349×10^{-22}	0.000016	0.000016	7.28448×10^{-19}	0.0016
	0.70	4.9×10^{-7}	4.9×10^{-7}	0	0.000049	0.000049	7.52165×10^{-19}	0.0049
	1.00	1.0×10^{-6}	1.0×10^{-6}	0	0.0001	0.0001	7.45389×10^{-19}	0.01

6 Conclusion

The aim is showing the existence and uniqueness solution of equation (2.9), moreover we discuss the stability of it by using the statued lemma and fixed point theorem. We differentiate from the case in [7] by existing the integral term ℑ□. We apply the ADM on the example (5.1) for different values of α, □ and noticed that :

1. From our studies by numerical table, we conclude that

- In the case α more than 0.4 the values of exact solutions and approximate solutions approach to be the same for small values of □.
- In the case of α = 0.01, when we evaluate the changes of the time, we notice at small values (□ = 0.01) time unit, the exact and approximate solutions tend to be the same (the error be suitable).
- If we fix the time □ and increase α, the exact and approximate solutions be the same(the error decrease).
- If we fix α and increase the time □ the exact and approximate solutions be so different

(the error increase).

- When the values of □ greater than 0.3 we conclude in general this method not suitable at small values of α.

2. From our studies by figures, we conclude that

- As □ increase to infinity, the error increase.
- For $\alpha \rightarrow \infty$ the exact and approximate solutions approach to be the same.(the error can be neglect).
- In figure (1), as the values of □, □ increases ,the corresponding values of exact and approximate solutions increases.

3. In figure 2, exist differentiate between the exact and approximate solutions which is large as we seen.
4. In figure 3, exist differentiate between the exact and approximate solutions which is noticed.
5. In figure 4, the difference between the exact and approximate solutions is started after □ = 0.6 and □ = 0.8.

(See [30-32]).

References

[1] Shavlakadze, N., Kharibegashvili, S., & Jokhadze, O. (2016). *An approximate solution of one class of singular integro-differential equations.* Transactions of A. Razmadze Mathematical Institute, 170(3), 420-.426

[2] Sharif, A. A., Hamoud, A. A., & Ghadle, K. P. (2020). Solving nonlinear integro-differential equations by using numerical techniques. Acta Universitatis Apulensis, 61, .45-53

[3] Shavlakadze, N., Odishelidze, N., & Criado-Aldeanueva, F. (2020). Exact solutions of some singular integro-differential equations related to adhesive contact problems of elasticity theory. Zeitschrift für angewandte Mathematik und Physik, 71(4), .1-9

[4] Matoog, R. T. (2016). Existence of at least one solution of singular Volterra-Hammerstein integral equation and its numerical solution. African Journal of Mathematics and Computer Science Research, 9(3), .15-23

[5] Alhazmi, S. E. A. (2017). New Model for Solving Mixed Integral Equation of the First Kind with Generalized Potential Kernel. Journal of Mathematics Research, 9(5), .18-29

[6] El-Borai, M. M. (2002). Some probability densities and fundamental solutions of fractional evolution equations. Chaos, Solitons & Fractals, 14(3), .433-440

[7] El-Borai, M. M. (2004). Semigroups and some nonlinear fractional differential equations. Applied Mathematics and Computation, 149(3), .823-831

[8] Abdou, M. A., El-Kojak, M. K., & Raad, S. A. (2013). Analytic and numeric solution of linear partial differential equation of fractional order. Global J. and Decision science. Ins.(USA), 13(3/10), .57-71

[9] El-Borai, M. M. (2000, August). Semigroups and incorrect problems for evolution equations. In 16th IMACS WORLD CONGRESS LAUSANNE.

[10] El-Borai, M. M., Moustafa, O. L., & Michael, F. H. (2001). On the correct formulation of a nonlinear differential equations in Banach space. International Journal of Mathematics and Mathematical Sciences, 26(7), .437-444

[11] Vladimirov, V. S. (1976). Generalized functions in mathematical physics. Moscow Izdatel Nauka.

[12] Feller, W. An introduction to probability theory and its applications. .1957

[13] Adomian, G. (1988). A review of the decomposition method in applied mathematics. Journal of mathematical analysis and applications, 135(2), .501-544

[14] Adomian, G. (2013). Solving frontier problems of physics: the decomposition method (Vol. 60). Springer Science & Business Media.

[15] Rach, R. (2012). A bibliography of the theory and applications of the Adomian decomposition method, 1961-2011. Kybernetes.

[16] Abbaoui, K., & Cherruault, Y. (1994). Convergence of Adomian's method applied to nonlinear equations. Mathematical and Computer Modelling, 20(9), .69-73

[17] Abbaoui, K., & Cherruault, Y. (1994). Convergence of Adomian's method applied to differential equations. Computers & Mathematics with Applications, 28(5), .103-109

[18] Abdulaziz, O., Noor, N. F. M., Hashim, I., & Noorani, M. S. M. (2008). Further accuracy tests on Adomian decomposition method for chaotic systems. Chaos, Solitons & Fractals, 36(5), .1405-1411

[19] Behiry, S. H., Hashish, H., El-Kalla, I. L., & Elsaid, A. (2007). A new algorithm for the decomposition solution of nonlinear differential equations. Computers & Mathematics with Applications, 54(4), .459-466

[20] El-Kalla, I. L. (2008). Convergence of the Adomian method applied to a class of nonlinear integral equations. Applied mathematics letters, 21(4), .372-376

[21] Panda, A., Santra, S., & Mohapatra, J. (2021). Adomian decomposition and homotopy perturbation method for the solution of time fractional partial integro-differential equations. Journal of Applied Mathematics and Computing, .1-18

[22] Chen, Y., Si, X., & Shen, B. (2011, July). Adomian method solution for flow of a viscoelastic fluid through a porous channel with expanding or contracting walls. In 2011 International Conference on Multimedia Technology (pp. 2393-2396). IEEE.

[23] Adesanya, S. O., & Ayeni, R. O. (2011). Existence and uniqueness result for couple stress bio-fluid flow model via Adomian decomposition method. International Journal of Nonlinear Science, 12(1), .16-24

[24] Adomian, G. (1986). Application of the decomposition method to the Navier-Stokes equations. Journal of Mathematical Analysis and Applications, 119(1-2), .340-360

[25] Shamshuddin, M. D., Mishra, S. R., Anwar Beg, O., & Kadir, A. (2019). Adomian decomposition method simulation of von Kármán swrling bioconvection nanofluid flow. Journal of Central South University, 26(10), .2797-2813

[26] Wazwaz, A. M. (2000). A new algorithm for calculating Adomian polynomials for nonlinear operators. Applied Mathematics and computation, 111(1), .33-51

[27] Biazar, J., Babolian, E., Kember, G., Nouri, A., & Islam, R. (2003). An alternate algorithm for computing Adomian polynomials in special cases. Applied Mathematics and Computation, 138(2-3), .523-529

[28] El-Borai, M. (2021). A nonlocal Cauchy problem for abstract Hilfer equation with fractional integrated semi groups. Turkish Journal of Computer and Mathematics Education (TURCOMAT), 12(12), .1640-1646

[29] El-Borai, M. M., & El-Nadi, K. E. S. (2019). On the solutions of Ill-posed Cauchy problems for some singular integro-partial differential equations. Global J. Math, 13, .899-905

[30] El-Borai, M. M., Exact solutions for some nonlinear fractional parabolic fractional partial differential equations, Journal of Applied Mathematics and Computation 206 (2008) 141- 153.

[31] El-Borai, M. M.& El-Nadi, K. E. S., Stochastic fractional models of the diffusion of covid-19, Advances in Mathematics: Scientific Journal 9 (2020), no.12, 10267-10280.

[32] El-Borai, M. M. & El-Nadi, K. E. S., A nonlocal Cauchy problem for abstract Hilfer equation with fractional integrated semi groups, Turkish Journal of Computer and Mathematics Education, Vol. 12 (2021), pp. 1640 – 1646